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A fixed point-based BDF method for solving differential Riccati equations

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Abstract

This paper describes an approach for solving differential Riccati equations (DRE), by means of the backward differentiation formula (BDF) and resolution of the corresponding implicit equation using Newton's method with a fixed point approach. The role and use of DRE is especially important in several applications such as optimal control, filtering, and estimation. The goodness of this new method is compared with respect to the so called Dieci method [L. Dieci, Numerical integration of the differential Riccati equation and some related issues, SIAM J. Numer. Anal. 29 (3) (1992) 781–815].

Keywords: Fixed point method; Differential Riccati equations; Backward differentiation formula methods; Algebraic Riccati equation

1. Introduction

This paper presents a methodology for solving the differential Riccati equation (DRE) based on backward differentiation formula and resolution of the corresponding implicit equation, using Newton's method with a fixed point approach.

Consider the differential Riccati equation (DRE)

$$\dot{X}(t) = A_{21}(t) + A_{22}(t)X(t) - X(t)A_{11}(t) - X(t)A_{12}(t)X(t), \qquad (1)$$
$$X(t_0) = X_0, \quad t_0 \leqslant t \leqslant t_f,$$

where $A_{11}(\cdot) \in R^{n\times n}$, $A_{22}(\cdot) \in R^{m\times m}$, $A_{12}(\cdot) \in R^{n\times m}$, $A_{21}(\cdot) \in R^{m\times n}$ and $X(\cdot) \in R^{m\times n}$.

The DRE arises in several applications, in particular in control theory, for example the time-invariant linear quadratic optimal control problem. In this case, the DRE has the following expression:

$$\dot{X}(t) = Q + A^T X + XA - XBR^{-1}B^T X, \qquad (2)$$

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where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $Q = Q^{\mathrm{T}} \in R^{n\times n}$ is positive semidefinite, and $R = R^{\mathrm{T}} \in R^{m\times m}$ is positive definite, representing respectively, the state matrix, the input matrix, the state weight matrix and the input weight matrix.

This work is concerned with the study and implementation of a method for solving the DRE (1) by numerical integration, using the BDF method. The result is an implicit scheme which solves the discrete version of the DRE (1) according to the property that the discretization of a polynomial differential equation reduces it to a polynomial algebraic equation of the same degree.

An example of this methodology appeared in [1], and it is known as DRESOL. This package is based on LSODE IVP software, developed by Hindmarsh [2]. Several methods have been implemented for solving the algebraic Riccati equation (ARE); however, in the context of stiff DREs, one of the better choices for solving the associated ARE, is to apply implicit schemes based on Newton's or quasi-Newton's methods. In both cases, at each iteration step a Sylvester equation has to be solved

$$G_{11}Y - YG_{22} = H, \tag{3}$$

where $G_{11} \in R^{n\times n}$, $G_{22} \in R^{m\times m}$ and $H \in R^{n\times m}$ change at each iteration if Newton's method is used.

A standard method for solving Eq. (3) is Bartels–Stewart algorithm [3]. By using this approach, matrices G_{11} and G_{22} are both reduced via orthogonal matrices U and V to real Schur form, obtaining the equivalent equation

$$(U^{\mathrm{T}}G_{11}U)(U^{\mathrm{T}}YV) - (U^{\mathrm{T}}YV)(V^{\mathrm{T}}G_{22}V) = U^{\mathrm{T}}HV. \tag{4}$$

Once the quasi-triangular problem (4) is solved for $U^{\mathrm{T}}YV$, then Y is easily recovered.

This paper is organized as follows. First, Section 2 describes the numerical integration method using BDF and Luca Dieci methodology as a starting point. Section 3 presents our approach for solving the DRE applying BDF and fixed point method. A theoretical study in terms of memory storage and flops requirements is included. A sequential implementation of the method has been carried out using standard linear algebra libraries such as basic linear algebra subroutines (BLAS) [4] and linear algebra PACKage (LAPACK) [5]. Section 4 presents the test battery and the experimental results. Finally, the conclusions and future work are outlined in Section 5.

2. Numerical integration by using BDF

The DRE occurs in several applications in different fields of science and engineering. It did not however receive enough attention in the numerical literature until the mid seventies. Since then, a great variety of methods have been proposed [6,7]. These methods can be grouped into seven classes [8]: Vectorized approach, Linearization approach [9–11], Chandrasekhar approach [12], Superposition methods [13,14], Matrix versions of ODE methods [1], Hamiltonian approach [15,16] and Magnus series [17].

Let the Riccati equation

$$\dot{X}(t) = F(t,X), \quad X(t_0) = X_0, \qquad t \in [t_0, t_{\mathrm{f}}], \tag{5}$$

where

$$F(t,X) = A_{21}(t) + A_{22}(t)X(t) - X(t)A_{11}(t) - X(t)A_{12}(t)X(t),$$

with $A_{11}(t) \in R^{n\times n}$, $A_{12}(t) \in R^{n\times m}$, $A_{21}(t) \in R^{m\times n}$, $A_{22}(t) \in R^{m\times m}$ and $X(t) \in R^{m\times n}$.

In the literature it is possible to find several methods for solving this equation. One of these methods is the well known backward differentiation formula (BDF) [18]. With a BDF scheme, the integration interval $t \in [t_0, t_{\mathrm{f}}]$ is divided so that the approximate solution in t_{k+1}, X_{k+1}, is obtained by solving the implicit equation

$$X_{k+1} - \sum_{j=0}^{p-1} \alpha_j X_{k-j} - h\beta F(t_{k+1}, X_{k+1}) = 0, \tag{6}$$

where coefficients α and β are those shown in Table 1.

We used $p = 5$ as BDF coefficients, see Table 1.