Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 188 (2007) 1467–1475
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

The tanh–coth method for solitons and kink solutions for nonlinear parabolic equations

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Abstract

The tanh–coth method is used to derive solitons and kink solutions for some of the well-known nonlinear parabolic partial differential equations. The equations include the Fisher equation, Newell–Whithead equation, Allen–Cahn equation, FitzHugh-Nagumo equation, Fisher's equation, and the Burgers–Fisher equation. The new tanh–coth approach provides abundant solitons and kink solutions in addition to the existing ones. The power of this manageable method is confirmed.

Keywords: Nonlinear parabolic equations; The tanh–coth method; Solitons; Kinks

1. Introduction

The tanh–coth method is used to study the some of the nonlinear parabolic PDEs [1–5] given by

$$u_t = u_{xx} + au + bu^n, \tag{1}$$

$$u_t = u_{xx} + a + be^{n\lambda}u, \tag{2}$$

$$u_t = u_{xx} - u(1-u)(a-u), \tag{3}$$

$$u_t = u_{xx} + auu_x, \tag{4}$$

and

$$u_t = u_{xx} + auu_x + ku(1-u), \tag{5}$$

where a, b, k and λ are constants.

Eq. (1) gives rise to three well known models. For $a = -4$, $b = 4$ and $n = 3$, Eq. (1) becomes the Allen–Cahn equation. The Allen-Cahn equation serves as a model for the study of phase separation in isothermal, isotropic, binary mixtures such as molten alloys [2,3]. If for $n = 3$ the coefficient b is replaced by $-b$, then Eq. (1) becomes

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.11.013

the Newell–Whitehead equation. The Newell–Whitehead equation describes the dynamical behavior near the bifurcation point for the Rayleigh–Bénard convection of binary fluid mixtures [4].

However, for $n = 2$ and $b = -a$, Eq. (1) reduces to the well-known Fisher's equation. The Fisher's equation combines diffusion with logistic nonlinearities [5]. Moreover, the Fisher's equation is a model for the propagation of a mutant gene, with u denoting the density of an advantageous. This equation is encountered in chemical kinetics and population dynamics. Because Eq. (1) represents at least three of the well-known parabolic equation, it will be named as the general parabolic equation.

Eq. (2) is a parabolic equation with exponential nonlinearity. Moreover, Eqs. (3)–(5) give the FitzHugh–Nagumo equation, the Burgers' equation, and the Burgers–Fisher equation respectively. The FitzHugh–Nagumo equation describes the dynamical behavior near the bifurcation point for the Rayleigh–Bénard convection of binary fluid mixtures [4].

Eqs. (1)–(5) arise in many scientific applications such as mathematical biology, quantum mechanics and plasma physics. It is well known that wave phenomena of plasma media and fluid dynamics are modelled by kink shaped tanh solution or by bell shaped sech solutions.

Several different approaches, such as Backlandtransformation, a bilinear form [6], inverse scattering method [7], Jacobi elliptic functions, and a Lax pair [8], have been used independently by which soliton and multi-soliton solutions are obtained. Ablowitz and Segur [7] implemented the inverse scattering transform method to handle the nonlinear equations of physical significance where soliton solutions and rational solutions were developed. Recently, the tanh method established in [9–11], and was effectively used in [12–21] among many others. The tanh method was subjected by some modifications using the Riccati equation.

The objectives of this work are twofold. Firstly, we seek to apply the tanh–coth method [22–24] to extend others' works. Secondly, we aim to add considerably new solitons and kinks solutions in addition to those derived in others' works to ascertain the power of the proposed scheme. In what follows, the method will be reviewed briefly because details can be found in [22–24].

2. The tanh–coth method

A PDE

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \tag{6}$$

can be converted to an ODE

$$Q(u, u', u'', u''', \ldots) = 0, \tag{7}$$

upon using a wave variable $\xi = (x - ct)$. Eq. (7) is then integrated as long as all terms contain derivatives where integration constants are considered zeros. Introducing a new independent variable

$$Y = \tanh(\mu\xi), \quad \xi = x - ct, \tag{8}$$

leads to the change of derivatives:

$$\begin{aligned} \frac{d}{d\xi} &= \mu(1 - Y^2)\frac{d}{dY}, \\ \frac{d^2}{d\xi^2} &= -2\mu^2 Y(1 - Y^2)\frac{d}{dY} + \mu^2(1 - Y^2)^2\frac{d^2}{dY^2}. \end{aligned} \tag{9}$$

The tanh–coth method [22–24] admits the use of the finite expansion

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{10}$$

where M is a positive integer, in most cases, that will be determined. Expansion (10) reduces to the standard tanh method [8–10] for $b_k = 0, 1 \leqslant k \leqslant M$. Substituting (10) into the ODE (7) results in an algebraic equation in powers of Y.