Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 188 (2007) 1513–1534
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Exact travelling wave solutions for the $(n+1)$-dimensional double sine- and sinh-Gordon equations

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Abstract

For the $(n+1)$-dimensional double sine- and sinh-Gordon equations, by using the approach of dynamical systems to the travelling wave solutions, in different regions of the parameter space, all possible bounded solutions (solitary wave solutions, kink and anti-kink wave solutions and periodic wave solutions, et al.) are obtained. Explicit exact parametric representations are given.

Keywords: Solitary wave solution; Kink and anti-kink wave solution; Periodic wave solution; Theory of bifurcation; Double sine- and sinh-Gordon equations

1. Introduction

It is well known that the exact travelling wave solutions of the sine- and sinh-Gordon equations have been extensively studied in the field of theoretical physics (see Narita [1], Kobayashi et al. [2] and cited reference therein).

In this paper, we consider the $(n+1)$-dimensional double sine-Gordon (DSG) equation

$$\sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial^2 \phi}{\partial t^2} = \alpha_1 \sin \phi + \alpha_2 \sin 2\phi \tag{1.1}$$

and double sinh-Gordon (DSHG) equation

$$\sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial^2 \phi}{\partial t^2} = \alpha_1 \sinh \phi + \alpha_2 \sinh 2\phi, \tag{1.2}$$

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.11.014

where n is a positive integer. For $\alpha_2 = 0$, Eq. (1.1) is a model of fluxon dynamics in Josephson junctions; dislocation dynamics in crystal lattices; vortex states in spin systems with an anisotropy created by an external magnetic field, etc (see Vitanov [3,4] and cited references therein).

We shall use the method of dynamical systems (see [5–10]) to find the exact travelling wave solutions of (1.1) and (1.2). By considering the dynamics of the travelling wave solutions determined by the travelling wave systems, we shall generally give all possible explicit exact travelling wave solutions for Eq. (1.1), (1.2) in the different parameter regions. More than 50 explicit exact parametric representations are obtained by using the elliptic functions and hyperbolic functions (see [11]).

To find travelling wave solutions for (1.1) and (1.2), we use the wave variable $\xi = \sum_{j=1}^{n}\mu_j x_j - ct$, where c is the propagating wave velocity. Then, (1.1) and (1.2) can be become the ordinary differential equation

$$\left(\sum_{j=1}^{n}\mu_j^2 - c^2\right)u_{\xi\xi} = \alpha_1 \sin u + \alpha_2 \sin 2u \tag{1.3}$$

and

$$\left(\sum_{j=1}^{n}\mu_j^2 - c^2\right)u_{\xi\xi} = \alpha_1 \sinh u + \alpha_2 \sinh 2u. \tag{1.4}$$

Denote that $A = \sum_{j=1}^{n}\mu_j^2 - c^2$, and we assume that $A \neq 0$.

(1.3) and (1.4) are equivalent to the following two systems:

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{\alpha_1}{A}\sin u\left(1 + \frac{2\alpha_2}{\alpha_1}\cos u\right) \tag{1.5}$$

and

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{\alpha_1}{A}\sinh u\left(1 + \frac{2\alpha_2}{\alpha_1}\cosh u\right). \tag{1.6}$$

The rest of this paper is organized as follows. In Section 2, we discuss the bifurcations of phase portraits of (1.5). In Section 3, corresponding to all bounded orbits given by Section 2, we give all possible exact explicit parametric representations of the travelling wave solutions for Eq. (1.1). In Section 4, we discuss the bifurcations of phase portraits of (1.6). In Section 5, corresponding to all bounded orbits given by Section 4, we give all possible exact explicit parametric representations of the travelling wave solutions for Eq. (1.2).

More recently, Liu et al. [12] considered the problem of to find exact travelling wave solutions of (1.1) with $n = 1$ by using the transformations $u = 2\arctan v$, $u = 2\arcsin v$ and $u = \arccos v$, respectively. They stated in [12] that "when the second transformation is considered, it can be easily proven that the DSG equation cannot be solved directly". In Section 6, we shall use the method of dynamical systems to show that the above three transformations are useful for finding the exact travelling wave solutions of (1.1). By considering the dynamics of the travelling wave solutions determined by the travelling wave systems under three different transformations, we shall generally give all possible explicit exact travelling wave solutions for (1.1) in the different parameter regions. Furthermore, we shall show that all these explicit exact travelling wave solutions in Section 6 are included in those we have given in Section 3.

2. Bifurcations of the phase portraits of (1.5)

We first consider the system (1.5), which has the first integral

$$H_1(u,y) = \frac{y^2}{2} + \frac{\alpha_1}{A}\left(\cos u + \frac{\alpha_2}{2\alpha_1}\cos 2u\right). \tag{2.1}$$

System (1.5) is periodic in u. Hence, the state (u,y) can be viewed on a phase cylinder $S^1 \times R$, where $S^1 = [-\pi,\pi]$ with $-\pi,\pi$ identified. Clearly, for $u \in [-\pi,\pi]$, if $\left|\frac{\alpha_1}{2\alpha_2}\right| < 1$ there exist five equilibrium points $O(0,0)$, $E_\pm(\pm u_1,0)$ and $B_\pm(\pm\pi,0)$, where $u_1 = \arccos\left(-\frac{\alpha_1}{2\alpha_2}\right)$. Otherwise, (1.5) has only three equilibrium points $O(0,0)$ and $B_\pm(\pm\pi,0)$.