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A modified method for a non-standard inverse heat conduction problem ☆

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Abstract

A non-standard inverse heat conduction problem is considered. Data are given along the line $x = 1$ and the solution at $x = 0$ is sought. The problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the data. In order to solve the problem numerically it is necessary to employ some regularization method. In this paper, we study a modification of the equation, where a fourth-order mixed derivative term is added. Error estimates for this equation are given, which show that the solution of the modified equation is an approximation of the heat equation. A numerical implementation is considered and a simple example is given. Some numerical results show the usefulness of the modified method.

Keywords: Ill-posed problem; Regularization; Inverse heat conduction; Error estimate

1. Introduction

We consider the following Cauchy problem for the heat equation in a quarter plane: given data along the line $x = 1$ determine the solution for $0 \leqslant x < 1$. More precisely, solve the Cauchy problem [1–3]:

$$\begin{cases} u_t + u_x = u_{xx}, & x > 0, \ t > 0, \\ u(x,0) = 0, & x \geqslant 0, \\ u(1,t) = g(t), & t \geqslant 0, \ u(x,t)|_{x\to\infty} \text{ bounded}. \end{cases} \tag{1.1}$$

We require that u be bounded as $x \to \infty$, to guarantee the uniqueness of the solution (see, e.g., [1,5]). Physically, this corresponds to a situation in which the end-point $x = 0$ is inaccessible, but for which one can make (internal) measurements at $x = 1$. The functions $g(\cdot)$ and $u(x,\cdot)$ are to be in $L^2(\mathbb{R})$, but we assume that g, u and other functions which will appear in the following vanish for $t < 0$. Of course, since g is measured, there will be measurement errors, and we would actually have as data some function $g_\delta \in L^2$, for which

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$$\|g_\delta - g\| = \|g_\delta - u(1,\cdot)\| \leqslant \delta, \tag{1.2}$$

where $\|\cdot\|$ denotes the L^2-norm and the constant $\delta > 0$ represents a bound on the measurement error. That is to say, practically, we have the following problem:

$$\begin{cases} u_t + u_x = u_{xx}, & x > 0,\ t > 0, \\ u(x,0) = 0, & x \geqslant 0, \\ u(1,t) = g_\delta(t), & t \geqslant 0,\ u(x,t)|_{x\to\infty} \text{ bounded.} \end{cases} \tag{1.3}$$

Note that, although we seek to recover u only for $0 \leqslant x < 1$, the problem specification includes the heat equation for $x > 1$, in this case we can easily get the solution of the direct problem. Since we can obtain u for $x > 1$, also $u_x(1,\cdot)$ is determined. Thus we can consider (1.3) as a Cauchy problem with appropriate Cauchy data $[u, u_x]$ given on the line $x = 1$.

The problem of solving (1.3) for $0 \leqslant x < 1$ is ill-posed: the solution (if it exists) does not depend continuously on the data g_δ (we can also understand this point from Section 2). Therefore, it is impossible to solve problem (1.3) using classical numerical methods. However, if we impose an a priori bound on the solution at $x = 0$, i.e.,

$$\|u(0,\cdot)\| \leqslant E, \tag{1.4}$$

where E is a finite positive constant, and, in addition, we choose an appropriate regularization method (which is usually employed to solve ill-posed problem, see, e.g., [6–9]), we can restore the stability of the solution of above ill-posed problem. Problem (1.3) has been studied by some authors using different method (see [1–4,10]). Some numerical methods have been developed for the general equation [11,12], however, in most cases the stability theory and convergence proofs have not been generalized accordingly. In this manuscript, a new regularization method, which is proposed as an alternative way of regularization methods for the heat equation, is given. Actually, we discuss the possibility of modifying (1.3) to obtain a stable approximation, i.e., we will investigate the following problem:

$$\begin{cases} v_t + v_x = v_{xx} - \mu^2 v_{xxtt}, & x > 0,\ t > 0, \\ v(x,0) = 0, & x \geqslant 0, \\ v(1,t) = g_\delta(t), & t \geqslant 0,\ v(x,t)|_{x\to\infty} \text{ bounded} \end{cases} \tag{1.5}$$

where the choice of μ is based on some a priori knowledge about the magnitude of the errors in the data g_δ. The idea of constructing the problem (1.5) comes from Eldén [13], in which he considered a standard inverse heat conduction problem. The similar method has been successfully used to solve a Cauchy problem for the Laplace equation [16].

We have not seen the problem (1.5) treated in the literature, and at present we are not deeply concerned with theoretical questions related to the problem (1.5). The main objective of this investigation is to find out how well (1.5) approximates (1.1) considered as a Cauchy problem in the space variable.

We give some error estimates in Section 3, which show that this problem can be used to approximate a solution of (1.1). Asymptotically, as the magnitude of data errors tend to zero, the error estimate tends to zero logarithmically.

An advantage of the modified equation (1.5) is that it can be discretized using standard techniques, e.g., finite differences. A numerical implementation is described in Section 4 and a simple example is given in Section 5. Some numerical results validate the usefulness of the modified method.

2. Some auxiliary results

Since we have defined all functions to be vanish for $t < 0$, the Fourier transform of the exact data g can be defined as below:

$$\hat{g}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-i\xi t}\, dt. \tag{2.1}$$