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Travelling wave solutions for modified Zakharov–Kuznetsov equation

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Abstract

By using the theory of bifurcations of dynamical systems to the modified Zakharov–Kuznetsov equation which describes wave propagation in isothermal multicomponent magnetized plasmas, numbers of solitary waves and periodic waves and kink waves are obtained. Under various parameter conditions, all explicit formulas of solitary waves solutions, periodic waves solutions and kink waves solutions are given.

Keywords: Solitary wave solution; Periodic wave solution; Kink and anti-kink wave solution; Theory of bifurcation; Modified Zakharov–Kuznetsov equation

1. Introduction

In Ref. [1], Das and co-workers investigated isothermal multicomponent magnetized plasma, and derived first Zakharov–Kuznetsov equation (ZK), then taking into account higher order nonlinear effect, derived the modified Zakharov–Kuznetsov equation (MZK)

$$\frac{\partial\phi}{\partial\tau}+[A\phi+D\phi^2]\frac{\partial\phi}{\partial\varsigma}+B\frac{\partial^3\phi}{\partial\varsigma^3}+\frac{\partial}{\partial\varsigma}\left[\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}\right]=0, \tag{1.1}$$

where A, B, D are plasma parameters, which describe the evolution of various solitary waves in isothermal multicomponent magnetized plasma. Das et al. discussed the soliton formation and propagation in nonlinear model (1.1), and compressive and rarefactive solitary wave solutions were obtained by the use of sech–tanh method; moreover, the spiky and explosive solitary wave solutions to Eq. (1.1) were also obtained by the use of the following traveling wave reduction:

$$\chi=l\xi+m\eta+n\varsigma-U\tau,\quad \phi(\xi,\eta,\tau)=\phi(\chi), \tag{1.2}$$

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where l, m, n and U are arbitrary constants, and usual boundary conditions

$$\phi \to 0, \quad \phi' \to 0, \quad \phi'' \to 0 \quad \text{as } |\chi| \to \infty. \tag{1.3}$$

However, the solutions obtained by Das et al. are very particular, because they employ special conditions (1.3). It is very natural to ask whether Eq. (1.1) has other solitary wave solutions except those shown in Ref. [1]. In this paper, we will consider bifurcation problem of solitary waves, kink waves and periodic waves for Eq. (1.1) by using the bifurcation theory of dynamical system [2,5,6,8,9]. Under fixed parameter conditions, all explicit formulas of solitary wave, kink wave and periodic wave solutions can be obtained.

To find the travelling wave solutions of (1.1), we first consider the traveling wave solutions in the form (1.2). By substituting the above traveling wave solutions (1.2) into (1.1) and integrating once, we have

$$n(n^2B + l^2 + m^2)\phi_{\chi\chi} = -\frac{1}{3}nD\phi^3 - \frac{1}{2}nA\phi^2 + U\phi - g, \tag{1.4}$$

where g is an integral constant. Eq. (1.4) is equivalent to the following two-dimensional system as follows:

$$\frac{d\phi}{d\chi} = y, \quad \frac{dy}{d\chi} = -\frac{D}{3k}\phi^3 - \frac{A}{2k}\phi^2 + \frac{U}{nk}\phi - \frac{g}{nk}, \tag{1.5}$$

where $n^2B + l^2 + m^2 = k$, which is a Hamiltonian system with Hamiltonian function

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{D}{12k}\phi^4 + \frac{A}{6k}\phi^3 - \frac{U}{2nk}\phi^2 + \frac{1}{nk}g. \tag{1.6}$$

All travelling wave solutions of (1.1) can be determined by the phase orbits of (1.5). Suppose that $\phi(\xi, \varsigma, \eta, \tau) = \phi(\chi)$ is a continuous solution of Eq. (1.1) for $\chi \in (-\infty, \infty)$ and $\lim_{\chi\to+\infty}\phi(\chi) = \alpha$, $\lim_{\chi\to-\infty}\phi(\chi) = \beta$. It is well kown that (i) If $\alpha = \beta$, $\phi(\chi)$ is called a solitary wave solution of (1.1), which corresponds to a homoclinic orbit of (1.5). (ii) If $\alpha \neq \beta$, $\phi(\chi)$ is called a kink or anti-kink wave solution of (1.1), which corresponds to a heteroclinic orbit of (1.5). (iii) Similarly, a periodic travelling solution of (1.1), which corresponds to a periodic orbit of (1.5) (see [3,4]). Thus, to investigate all bifurcations of solitary waves, kink waves and periodic waves of Eq. (1.1), we should find all periodic annuli, homoclinic and heteroclinic orbits of (1.5) depending on the parameter space of this system. The bifurcation theory of dynamical systems plays an important role in our study.

This paper is organized as follows. In Section 2, we give the bifurcation set and phase portraits of (1.1). In Section 3, we show all explicit formulae of solitary wave solutions and kink (or anti-kink) wave solutions under given parameter conditions. In Section 4, we show all explicit formulae of periodic wave solutions under given parameter conditions. Our study results give rise to complete description of all traveling waves of (1.1) and contain results in [1] as special examples.

2. Bifurcations of phase portraits of (1.1)

In this section, we consider bifurcation set and phase portraits of (1.1). Obviously, on the (ϕ, y)-phase plane, the abscissas of equilibrium points of system (1.5) are the zeros of $f(\phi) = \phi^3 + \frac{3A}{2D}\phi^2 - \frac{U}{nD}\phi + \frac{g}{nD}$. Let (ϕ_c, y) be an equilibrium point of (1.5). At this point, the determinant of the linearized system of (1.5) has the form $J(\phi_c, 0) = \frac{D}{3k}f'(\phi_c)$. By using the bifurcation theory of planar dynamical system, we know that if $J(\phi_c, 0) > 0$ (or <0), then equilibrium point $(\phi_c, 0)$ is a center (or saddle point); if $J(\phi_c, 0) = 0$ and the Poincaré index of $(\phi_c, 0)$ is zero, then the equilibrium point $(\phi_c, 0)$ is a cusp point.

Case I. $\frac{U}{nk} = 0$.

Making the transformation $\sqrt{\frac{D}{3k}}\chi = \rho$, $y \to \sqrt{\frac{D}{3k}}y$ for $Dk > 0$, or $\sqrt{-\frac{D}{3k}}\chi = \rho$, $y \to \sqrt{-\frac{D}{3k}}y$ for $Dk < 0$, then (1.5) becomes the following two-dimensional system:

$$\frac{d\phi}{d\rho} = y, \quad \frac{dy}{d\rho} = \pm(\phi^3 + p\phi^2 + q), \tag{2.1}$$

where $p = \frac{3A}{2D}$, $q = \frac{3g}{nD}$ and when $Dk < 0$ (or >0), the sign in the right-hand side of second equation of ((2.1) is "+" (or "−")). Eq. (2.1) is a Hamiltonian system with Hamiltonian function