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The convergence theorem for a family deformed Chebyshev method in Banach space

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Abstract

Under the γ condition, we establish the convergence theorem of a family deformed Chebyshev method in Banach space which is used to solve the nonlinear operator equation. Finally, two examples are provided to show the application of the theorem.

Keywords: Chebyshev method; Banach space; Nonlinear operator equation; Convergence theorem

1. Introduction

Solving the nonlinear operator equation is a important issue in the engineering and technology field. In this study, we consider to establish the convergence theorem of a family deformed Chebyshev iterative in Banach space which is used to solve the nonlinear operator equation

$$F(x) = 0, \tag{1.1}$$

where F is defined on an open convex Ω of a Banach space X with values in a Banach space Y.

There are kinds of methods to find a solution of (1.1). Iterative methods are often used to solve this problem (see [1]). If we use the famous Newton's method, we can do as

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad (n \geqslant 0)\ (x_0 \in \Omega). \tag{1.2}$$

Under the reasonable hypothesis, Newton's method is two orders convergence.

To improve the convergence order, many deformed methods have been present. The Chebyshev method is three orders. The definition is as follows:

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$$x_{n+1} = x_n - \left[I + \frac{1}{2}L_F(x_n)\right]F'(x_n)^{-1}F(x_n). \tag{1.3}$$

In (1.3), $L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x)$, $x \in \Omega$. Refs. [2–7] established the convergence theorems for the Chebyshev method. In (1.3), every step need compute two orders derivatives of the function. The computing cost will be high. To avoid the computation of $F''(x)$, and maintain the high order convergence, they present the deformed Chebyshev method as follows:

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ H(x_n, y_n) = 2F'(x_n)^{-1}\left[F'\left(x_n + \frac{1}{2}(y_n - x_n)\right) - F'(x_n)\right], \\ x_{n+1} = y_n - \frac{1}{2}H(x_n, y_n)(y_n - x_n). \end{cases} \tag{1.4}$$

Refs. [8,9] established the convergence theorem for the deformed Chebyshev method. In the study, we present a deformed Chebyshev method family. The method is defined as follows:

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ H(x_n, y_n) = 2F'(x_n)^{-1}\left[F'\left(x_n + \frac{1}{2}(y_n - x_n)\right) - F'(x_n)\right], \quad \lambda \in [0,1], \\ x_{n+1} = y_n - \frac{1}{2}H(x_n, y_n)[I - \lambda H(x_n, y_n)](y_n - x_n). \end{cases} \tag{1.5}$$

We establish the convergence theorem under γ-conditions. The convergence order is three. If we select reasonable λ to some issues, the convergence order is four. To apply the result, in Example 1 when $\lambda = \frac{23}{24}$, we get the convergence is of four orders.

2. Main results

Denote $g(t) = \beta - t + \frac{\gamma t^2}{1-\gamma t}$, where $t \in [0, \frac{1}{\gamma})$, β, γ are the positive real numbers. Let $\alpha = \beta\gamma$, then, when $\alpha \leqslant 3 - 2\sqrt{2}$, g has two positive real roots $t^* = \frac{1+\alpha-\sqrt{(1+\alpha)^2-8\alpha}}{4\gamma}$, $t^{**} = \frac{1+\alpha+\sqrt{(1+\alpha)^2-8\alpha}}{4\gamma}$, and satisfies

$$\beta \leqslant t^* \leqslant \left(1 + \frac{1}{\sqrt{2}}\right)\beta \leqslant \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\gamma} \leqslant t^{**} \leqslant \frac{1}{2\gamma}.$$

Let $t_0 = 0$, the sequence $\{t_n\}$, $\{s_n\}$ are generated by the following formula:

$$\begin{cases} s_n = t_n - g'(t_n)^{-1}g(t_n), \\ H_g(t_n, s_n) = 2g'(t_n)^{-1}\left[g'\left(t_n + \frac{1}{2}(s_n - t_n)\right) - g'(t_n)\right], \quad \lambda \in [0,1], \\ t_{n+1} = s_n - \frac{1}{2}H_g(t_n, s_n)[1 - \lambda H_g(t_n, s_n)](s_n - t_n). \end{cases} \tag{2.1}$$

Definition 2.1. If $F(x)$ exists three orders Fréchet derivatives in Ω, $x_0 \in \Omega$, $F'(x_0)^{-1}$ exists and

$$\|F'(x_0)^{-1}F(x_0)\| \leqslant \beta, \quad \|F'(x_0)^{-1}F''(x_0)\| \leqslant 2\gamma,$$
$$\|F'(x_0)^{-1}F'''(x)\| \leqslant \frac{6\gamma^2}{(1-\gamma\|x-x_0\|)^4} = g'''(\|x-x_0\|), \quad x \in \Omega,\ \|x-x_0\| \leqslant \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\gamma}.$$

Then we define that $F(x)$ satisfies γ-condition (see [10]).

Lemma 2.1. *If F satisfies γ-condition, $\|x - x_0\| < \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\gamma}$. Then*

(a) $\|F'(x_0)^{-1}F''(x)\| \leqslant g''(\|x-x_0\|)$,
(b) $F'(x)^{-1}$ *exists, and* $\|F'(x)^{-1}F'(x_0)\| \leqslant -\frac{1}{g'(\|x-x_0\|)}$.