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Newton's method for the quadratic matrix equation

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Abstract

Under suitable condition, a quadratic matrix equation is equivalent to a nonlinear matrix equation. We apply Newton's method to the nonlinear matrix equation for computing the numerical solution of a class of quadratic matrix equations. Convergence theorems are derived and numerical results are also given.

Keywords: Quadratic matrix equation; Nonlinear matrix equation; Iteration method; Newton's method; Convergence theorem

1. Introduction

Quadratic matrix equations (**QME**) arises in many areas of scientific computing and engineering applications. The most famous quadratic matrix equation is the *algebraic Riccati equation* (**ARE**), which occurs in a variety of problems [6,12,18]. In this paper, we mainly study another class of quadratic matrix equations:

$$\mathcal{Q}(X) = AX^2 + BX + C = 0, \tag{1}$$

where A, B and $C \in \mathbb{R}^{n\times n}$ and

$$(x^{\mathrm{T}}Bx)^2 > 4(x^{\mathrm{T}}Ax)(x^{\mathrm{T}}Cx), \quad \forall x \neq 0. \tag{2}$$

These equations are connected with the overdamped quadratic eigenvalue problem in the analysis of damped structural systems and vibration problem [7,9,16,17].

The most popular solver for the nonlinear system is the well-known Newton method. As we have known, under some suitable conditions Newton's method can achieve quadratic convergence speed. Davis [7,8] considered Newton's method to the quadratic matrix (1) in detail. In order to improve the global convergence of Newton's method, Higham and Kim [16] incorporated exact line search into Newton's method. Other different techniques for analysis and solution to the quadratic matrix equation (1) and other nonlinear problems can also been found (see [1,3–5,10,11,14,15,22,24], for example). Besides, as special cases of the quadratic matrix equation, several linear matrix equations have been recently studied in [19–21].

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In this paper, we study another different technique to solve the **QME** (1) with the condition (2). It is readily seen that if the quadratic matrix equation (1) has a nonsingular solution S, S is also the solution of the nonlinear matrix equation (**NME**)

$$\mathscr{F}(X) = AX + CX^{-1} + B = 0. \tag{3}$$

We apply Newton's method to the nonlinear matrix equation (3) for computing the *dominant solvent* and *the minimal solvent* ([15], definition 5) of the quadratic matrix equation (1). Motivation and effectiveness of this technique are given in the following. Because the minimal solvent of the **QME** (1) is also the dominant solution of the **NME** (3) (provided that the minimal solvent is nonsingular). Therefore, without loss of generality, throughout this paper we shall only focus on the dominant solvent.

The paper is arranged as follows. In Section 2, we discuss the scalar case to give an intuitionistic interpretation and analysis of our technique. Section 3 is mainly devoted to Newton's method to the **NME** (3). Under some suitable conditions, We derive the local convergence theorem and semilocal convergence theorem for our new Newton's method in Section 4. And finally, in Section 5, numerical results are presented to show the feasibility and effectiveness of our new method.

2. Scalar case

In this section, we consider briefly a scalar quadratic equation $q(x) = ax^2 + bx + c = 0$ $(a > 0)$ and its corresponding equivalent nonlinear equation $f(x) = ax + \frac{c}{x} + b = 0$ $(a > 0, x \neq 0)$. Obviously, the overdamped condition (2) is degenerative to be $b^2 - 4ac > 0$, which is just the judgement condition of the scalar quadratic equation having two different roots. Denote the two roots by α, β, and $|\alpha| > |\beta| > 0$. Noting that $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$ and doing some concrete calculations, we obtain the following conclusions:

(1) The nonlinear function $f(x) = ax + \frac{c}{x} + b$ $(a > 0)$ consists of the linear item $ax + b$ and nonlinear item $\frac{c}{x}$. When x is large, the linear part is dominant. Hence, if $|\alpha|$ is large, our strategy is feasible. On the contrary, when $|\alpha|$ is close to zero, our method is on a poor way.
(2) We consider their derivatives at α. It is easy to obtain that $q'(\alpha) = 2a\alpha + b = a(\alpha - \beta)$ and $f'(\alpha) = a - \frac{c}{\alpha^2} = a\left(1 - \frac{\beta}{\alpha}\right)$. Therefore, provided that $|\alpha - \beta| \gg 1$, then $|q'(\alpha)| > f'(\alpha)$. If we apply Newton's method to the two scalar equation, our technique is more effective.

In the last place, we will take $q(x) = x^2 - 5x - 6 = 0$ for an example to illustrate the above two points. Easily, we can calculate that $f(x) = x - \frac{6}{x} - 5 = 0$ and $\alpha = 6$, $\beta = -1$, $|\alpha - \beta| = 7$. First, the above two points can also be seen from Fig. 1. Numerical results also further confirm our conclusions. We apply Newton's method to $f(x) = 0$ and $q(x) = 0$. Set the error $r = |x^2 - 5x - 6|$ and tolerance $\epsilon = 10^{-8}$. For different initial values, We list their corresponding iteration steps in Table 1.

3. Newton's method

In what follows, the F-(or Frechet-) derivative of mapping $\mathscr{F} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by (3) at X is needed. Based on the definition of F-differentiable and some simple calculation, we obtain that if the matrix X is nonsingular, then the mapping $\mathscr{F}$ is F-differentiable at X and

$$\mathscr{F}'_X(Z) = AZ - CX^{-1}ZX^{-1}. \tag{4}$$

Besides, the second-derivative of $\mathscr{F}$ at X is also well defined and

$$\mathscr{F}''_X(Y,Z) = CX^{-1}YX^{-1}ZX^{-1} + CX^{-1}ZX^{-1}YX^{-1}.$$

Obviously, the F-derivatives of mapping $\mathscr{F}$ and their inverse are nonlinear operators. Then, applying Newton's method to the nonlinear matrix equation (3), we have

$$X_{k+1} = X_k - (\mathscr{F}'_{X_k})^{-1}(\mathscr{F}(X_k)), \quad k = 0, 1, \ldots \tag{5}$$