Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 185 (2007) 266–271
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

The block smoothing method for estimating the spectral radius of a nonnegative matrix

Zhi-Ming Yang

Mathematics and Information College, Gansu Lianhe University, Lanzhou 730000, PR China

Abstract

Utilizing the block matrix, a new estimate for the spectral radius of a nonnegative matrix is presented. A series of new matrices are derived that preserve the spectral radius while their minimum row sums are on the increase and the maximum row sums on the decrease. Numerical examples are provided to illustrate the effectiveness of this approach.

Keywords: Nonnegative matrix; Block matrix; Spectral radius; Smoothing method

1. Introduction

Let A be a nonnegative matrix of order n with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$. The set of $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ is called the spectrum of A. For an $n\times n$ nonnegative matrix A, a fundamental matrix problem is to estimate spectral radius, $\rho(A)=\max_i|\lambda_i|$. So it is interesting to develop methods giving rise to bounds for $\rho(A)$. It is well known that for such a matrix A, the following inequality holds ([1,2]):

$$\min_i R_i = r(A) \leqslant \rho(A) \leqslant R(A) = \max_i R_i, \tag{1.1}$$

where $R_i=\sum_{j=1}^n a_{ij}$, $1\leqslant i\leqslant n$.

This result bears a great idea for estimating the spectral radius by using the elements of A in a simple way. The question arising is how to get sharper bounds by increasing the minimum row sum and decreasing the maximum row sum. This paper concentrates on developing a new method of getting sharper bounds and even a fairly good approximate value for $\rho(A)$, utilizing only the elements of a series of block matrices.

2. Partitioning matrix and notation

In this section, we introduce the method for partitioning a matrix and provide some notation, which will be used later.

E-mail addresses: yzhming0809@163.com, yangzm0202@msn.com

0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.06.108

Assume that n is the product of k numbers: $n_1, n_2, \ldots, n_k$, that is

$$n = n_1 \times n_2 \times \cdots \times n_k.$$

Without loss of generality, we suppose that $n_i (i = 1,2,\ldots,k)$ are prime numbers and can be ordered:

$$n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k.$$

In this case, an $n \times n$ nonnegative matrix A can be partitioned into as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n_1} \\ A_{21} & A_{22} & \cdots & A_{2n_1} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n_1 1} & A_{n_1 2} & \cdots & A_{n_1 n_1} \end{bmatrix},$$

where $A_{ij}\,(i,j = 1,2,\ldots,n_1)$ are matrices of order $\frac{n}{n_1}$. Now we add the elements which are at the same position of each A_{ij} and then divided by n_1^2, thus, a small matrix of order $n_2 \times n_3 \times \cdots \times n_k$ can be obtained, we denote it as

$$A^{(1)} = \frac{1}{n_1^2} \sum_{i,j=1}^{n_1} A_{ij} = (a_{ij}^{(1)}),$$

and it is easy to prove the following inequalities:

$$R(A^{(1)}) \leqslant \frac{1}{n_1} R(A), \quad r(A^{(1)}) \geqslant \frac{1}{n_1} r(A), \tag{2.1}$$

where $R(A^{(1)}) = \max_i R_i(A^{(1)}), r(A^{(1)}) = \min_i R_i(A^{(1)}), R_i(A^{(1)}) = \sum_{j=1}^{\frac{n}{n_1}} a_{ij}^{(1)}$ for $i = 1, 2, \ldots, \frac{n}{n_1}$. Analogously, we partition $A^{(1)}$ into

$$A^{(1)} = \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & \cdots & A_{1n_2}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} & \cdots & A_{2n_2}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n_2 1}^{(1)} & A_{n_2 2}^{(1)} & \cdots & A_{n_2 n_2}^{(1)} \end{bmatrix},$$

where $A_{ij}^{(1)}(i,j = 1,2,\ldots,n_2)$ are $\frac{n}{n_1 n_2} \times \frac{n}{n_1 n_2}$ matrices. Again we add the elements at the same position of each $A_{ij}^{(1)}$ and divided by n_2^2, then we can get a matrix of order $n_3 \times\ n_4 \times \cdots \times n_k$, and it could be partitioned into an $n_3 \times n_3$ block matrix, i.e.,

$$A^{(2)} = \frac{1}{n_2^2} \sum_{i,j=1}^{n_2} A_{ij}^{(1)} = (a_{ij}^{(2)}) = (A_{ij}^{(2)})_{n_3 \times n_3},$$

where $A_{ij}^{(2)}(i,j = 1,2,\ldots,n_3)$ are $\frac{n}{n_1 n_2 n_3} \times \frac{n}{n_1 n_2 n_3}$ matrices.

Let $R(A^{(2)}) = \max_i R_i(A^{(2)}), r(A^{(2)}) = \min_i R_i(A^{(2)}), R_i(A^{(2)}) = \sum_{j=1}^{\frac{n}{n_1 n_2}} a_{ij}^{(2)}$ for $i = 1, 2, \ldots, \frac{n}{n_1 n_2}$, then the following inequalities hold:

$$R(A^{(2)}) \leqslant \frac{1}{n_2} R(A^{(1)}), \quad r(A^{(2)}) \geqslant \frac{1}{n_2} r(A^{(1)}). \tag{2.2}$$

At last, we will get an $n_k \times n_k$ matrix

$$A^{(k-1)} = \frac{1}{n_{k-1}^2} \sum_{i,j=1}^{n_{k-1}} A_{ij}^{(k-2)} = (a_{ij}^{(k-1)})_{n_k \times n_k}$$

and two inequalities:

$$R(A^{(k-1)}) \leqslant \frac{1}{n_{k-1}} R(A^{(k-2)}), \quad r(A^{(k-1)}) \geqslant \frac{1}{n_{k-1}} r(A^{(k-2)}), \tag{2.3}$$

where $R(A^{(k-1)}) = \max_i R_i(A^{(k-1)}), r(A^{(k-1)}) = \min_i R_i(A^{(k-1)})$ and $R_i(A^{(k-1)}) = \sum_{j=1}^{n_k} a_{ij}^{(k-1)}$ for $i = 1,2,\ldots,n_k$.