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A numerical treatment for singularly perturbed differential equations with integral boundary condition

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Abstract

We consider a uniform finite difference method on Shishkin mesh for a quasilinear first order singularly perturbed boundary value problem (BVP) with integral boundary condition. We prove that the method is first order convergent except for a logarithmic factor, in the discrete maximum norm, independently of the perturbation parameter. The parameter uniform convergence is confirmed by numerical computations.

Keywords: Finite difference; Singular perturbation; Shishkin mesh; Integral boundary condition; Error estimates

1. Introduction

In this paper we consider the following singular perturbation problem (BVP) with integral boundary condition

$$\varepsilon u' + f(t,u) = 0, \quad t \in I = (0,T], \; T > 0, \tag{1.1}$$

$$u(0) = \mu u(T) + \int_0^T b(s)u(s)\,\mathrm{d}s + d, \tag{1.2}$$

where $0 < \varepsilon \leqslant 1$ is the perturbation parameter, μ and d are given constants. $b(t)$ and $f(t,u)$ are assumed to be sufficiently continuously differentiable functions in $\bar{I} = I \cup \{t = 0\}$ and $\bar{I} \times \mathbb{R}$ respectively and moreover

$$\frac{\partial f}{\partial u} \geqslant \alpha > 0.$$

Note that the boundary condition (1.2) includes periodic and initial conditions as special cases. For $\varepsilon \ll 1$ the function $u(t)$ has a boundary layer of thickness $O(\varepsilon)$ near $t = 0$ (see Section 2).

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Differential equations with integral boundary conditions constitute a very interesting and important class of problems and have been studied for many years. For a discussion of existence and uniqueness results and for applications of problems with integral boundary conditions see, [7–10,12,16,17] and the references therein. In [9,16,17] have been considered some approximating aspects of this kind of problems. But designed in the above-mentioned papers algorithms are only concerned with the regular cases (i.e. when the boundary layers are absent).

Differential equations with a small parameter ε multiplying the highest order derivative terms are said to be singularly perturbed and normally boundary layers occur in their solutions. The numerical analysis of singular perturbation cases has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid changes within very thin layers near the boundary or inside the problem domain [4,5,13–15]. It is well known that standard numerical methods for solving such problems are unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to these problems, whose accuracy does not depend on the parameter value ε, i.e. methods that are convergence ε-uniformly. For the various approaches on the numerical solution of differential equations with steep, continuous solutions we may refer to the monographs [14,5,15].

In this present paper, we analyze a finite difference scheme on a special piecewise uniform mesh (a Shishkin mesh) for the numerical solution of the problem with integral boundary conditions (1.1) and (1.2). In Section 2, we state some important properties of the exact solution. The derivation of the difference scheme and mesh introduction have been given in Section 3. In Section 4, we present the error analysis for approximate solution. The method comprises a special non-uniform mesh, which is fitted to the initial layer and constructed a priori in function of sizes of parameter ε, the problem data and the number of corresponding mesh points. Uniform convergence is proved in the discrete maximum norm. In Section 5, we formulate the iterative algorithm for solving the discrete problem and give the illustrative numerical results. The technique to construct discrete problem and error analysis for approximate solution are similar to those in [1–3].

Henceforth, C and c denote the generic positive constants independent of ε and of the mesh parameter. Such a subscripted constant is also independent of ε and mesh parameter, but whose value is fixed.

2. The continuous problem

Lemma 2.1. *Assume that the first derivative of $f(t,u)$ in u is uniformly bounded. Moreover*

$$p(\varepsilon) = 1 - \mu A^+ - b^* B^+ \geqslant c_0 > 0, \tag{2.1}$$

where

$$A^+ = \begin{cases} 0, & \mu \leqslant 0, \\ \varepsilon e^{-\alpha T/\varepsilon}, & \mu > 0, \end{cases} \qquad B^+ = \begin{cases} 0, & b^* \leqslant 0, \\ \alpha^{-1}\varepsilon(1 - e^{-\alpha T/\varepsilon}), & b^* > 0, \end{cases}$$
$$b^* = \max_{\bar{I}} b(x).$$

Then the following estimates hold:

$$\|u\|_\infty \leqslant C_0, \tag{2.2}$$

where

$$C_0 = c_0^{-1}(|\mu| + \|b\|_1)\alpha^{-1}\|F\|_\infty + c_0^{-1}|d|, \quad \|b\|_1 = \int_0^T |b(t)|\,dt,$$
$$F(t) = f(t,0)$$

and

$$|u'(x)| \leqslant C\left\{1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}}\right\}, \quad t \in \bar{I} \tag{2.3}$$

provided $|\partial f/\partial t| \leqslant C$ *for* $t \in \bar{I}$ *and* $|u| \leqslant C_0$.