Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 187 (2007) 883–892
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

# Convergence of three-step iterations for asymptotically nonexpansive mappings

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**Abstract**

In this paper, weak and strong convergence theorems of three-step iterations are established for asymptotically nonexpansive mappings in Banach spaces. The results obtained in this paper extend and improve the recent ones announced by Xu and Noor [B.L. Xu, M.A. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 267 (2002) 444–453], Suthep Suantai [S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 311 (2005) 506–517], Weerayuth Nilsrakoo and Satit Saejung [W. Nilsrakoo, S. Saejung, A new three-step fixed point iteration scheme for asymptotically nonexpansive mappings, Appl. Math. Comput., in press, doi:10.1016/j.amc.2006.01.063] and others.


*Keywords:* Asymptotically nonexpansive mapping; Three-step iterations; Uniformly convex Banach space; Convergence theorem

## 1. Introduction

Let $X$ be a real Banach space and $C$ be a nonempty subset of $X$. A mapping $T: C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \geqslant 1$ and $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\|$$

for all $x, y \in C$ and all $n \geqslant 1$. The mapping $T$ is called uniformly L-Lipschitzian if there exists some positive constant $L$ such that

$$\|T^n x - T^n y\| \leqslant L \|x - y\|,$$

for all $x, y \in C$ and $n \geqslant 1$.

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[1] This research is supported by the Higher Education Commission, Pakistan, through grant No: 1-28/HEC/HRD/2005/90.

0096-3003/$ - see front matter 
doi:10.1016/j.amc.2006.09.008

Since Schu's results [1,2], the modified Mann and Ishikawa iterative schemes have been studied extensively by various authors to approximate fixed points of asymptotically nonexpansive mappings (see [1–10]).

In 2000, Noor [11] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [15] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [15] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge et al. [16] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [15] and applied these schemes to obtain new splitting-type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Xu and Noor [12] introduced and studied a three-step scheme to approximate fixed point of asymptotically nonexpansive mappings in a Banach space, whereas Cho et al. [13] extend their schemes to the three-step iterative scheme and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Very recently, Nilsrakoo and Saejung [17] defined a new three-step iterations which is an extension of Noor iterations and gave some weak and strong convergence theorems of the modified Noor iterations for asymptotically nonexpansive mappings in Banach space. It is clear that the modified Noor iterations include Mann iterations, Ishikawa iterations and original Noor iterations as special cases. Consequently results obtained in this paper can be considered as a refinement and improvement of the previously known results. The scheme is defined as follows.

Let $X$ be a normed linear space, $C$ be a nonempty convex subset of $X$, and $T : C \to C$ be a given mapping.

**Algorithm 1.** For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + \gamma_n T^n x_n + (1 - \alpha_n - \beta_n - \gamma_n)x_n, \quad n \geqslant 1, \end{aligned} \tag{1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are approximate sequences in $[0,1]$.

The iterative schemes (1) are called the three-step mean value iterations. If $\gamma_n \equiv 0$, then (1) reduces to the modified Noor iterations defined by Suantai [14] as follows:

**Algorithm 2.** For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n)x_n, \quad n \geqslant 1, \end{aligned} \tag{2}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are approximate sequences in $[0,1]$.

If $c_n = \beta_n = \gamma_n \equiv 0$, then (1) reduces to Noor iterations defined by Xu and Noor [12] as follows:

**Algorithm 3.** For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n, \\ y_n &= b_n T^n z_n + (1 - b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1, \end{aligned}$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are approximate sequences in $[0,1]$.

If $a_n = c_n = \beta_n = \gamma_n \equiv 0$, then (1) reduces to the modified Ishikawa iterative scheme as follows:

**Algorithm 4.** For a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative scheme

$$\begin{aligned} y_n &= b_n T^n x_n + (1 - b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1, \end{aligned}$$

where $\{\alpha_n\}$ and $\{b_n\}$ are approximate sequences in $[0,1]$.