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Modified Chebyshev's method free from second derivative for non-linear equations

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Abstract

In this paper, we present some new modifications of Chebyshev's method free from second derivative. The order of convergence of the new methods is three. Per iteration the methods require two evaluations of the function and one evaluation of its first derivative. The convergence of the new methods for the case of multiple roots is also discussed. Numerical examples show that the new methods have definite practical utility.

Keywords: Chebyshev's method; Non-linear equations; Newton's method; Root-finding; Iterative method

1. Introduction

Solving non-linear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find the roots of non-linear equations $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D is a scalar function.

Newton's method (NM) for a single non-linear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This is an important and basic method [1], which converges quadratically.

Chebyshev's method [2,3], known as its order of convergence three, is written as

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{f''(x_n)f(x_n)}{f'(x_n)^2}\right)\frac{f(x_n)}{f'(x_n)}. \tag{2}$$

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However, Chebyshev's method depends on the second derivatives in computing process and its practical application is restricted rigorously, so that Newton's method is frequently used to solve non-linear equations because of higher computational efficiency.

For this reason, a family of iterative methods with free second derivative is developed

$$x_{n+1} = x_n - \left(1 - \frac{1}{2\theta}\frac{f'(y_n) - f'(x_n)}{f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{3}$$

by replacing the second derivative of Chebyshev's method with a finite difference between first derivatives, i.e.

$$f''(x_n) \simeq \frac{f'(y_n) - f'(x_n)}{y_n - x_n},$$

where $\theta \in \mathbb{R}$, $\theta \neq 0$ and $y_n = x_n - \theta f(x_n)/f'(x_n)$. The cases $\theta = 1/2$, $\theta \in (0, 1]$ and $\theta < 0$ are considered in [4–6], respectively. These methods are important and interesting because they do not require the second derivative although they can converge cubically.

Recently, another approach is used to remove the second derivative from Halley's method and some third-order iterative methods free from second derivative are obtained [7].

Here, we will apply the approach in [7] to Chebyshev's method and obtain a family of modifications of Chebyshev's method free from second derivative. Analysis of convergence shows that this family of methods is cubically convergent for the case of simple roots. Per iteration these methods require two evaluations of the function and one of its first derivative. The convergence for the the case of multiple roots is also considered. Numerical examples show that the new methods have the definite practical utility.

2. The methods

Let $y_n = x_n - \theta f(x_n)/f'(x_n)$, where θ is a nonzero real parameter. We consider Taylor expansion of $f(y_n)$ about x_n

$$f(y_n) \simeq f(x_n) + f'(x_n)(y_n - x_n) + \frac{1}{2}f''(x_n)(y_n - x_n)^2,$$

which implies

$$f(y_n) \simeq (1-\theta)f(x_n) + \frac{1}{2}\theta^2\frac{f''(x_n)f(x_n)^2}{f'(x_n)^2}. \tag{4}$$

We can now approximate [7]

$$\frac{1}{2}\frac{f''(x_n)f(x_n)}{f'(x_n)^2} \simeq \frac{f(y_n) + (\theta - 1)f(x_n)}{\theta^2 f(x_n)}. \tag{5}$$

Using (5) in (2), we obtain

$$x_{n+1} = x_n - \frac{f(y_n) + (\theta^2 + \theta - 1)f(x_n)}{\theta^2 f'(x_n)}, \tag{6}$$

where $\theta \in \mathbb{R}$, $\theta \neq 0$ and $y_n = x_n - \theta f(x_n)/f'(x_n)$. In Section 3, we will prove that the methods defined by (6) are cubically convergent for any nonzero real number θ (for the case of simple roots).

If considering the different values of the parameter θ in (6), we can obtain a family of Chebyshev-type methods that includes, as particular cases, the following ones:

1. For $\theta = 1$, a third-order method, called Potra–Pták method [8] (PPM), is obtained

$$x_{n+1} = x_n - \frac{f(x_n - f(x_n)/f'(x_n)) + f(x_n)}{f'(x_n)}. \tag{7}$$