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Non-standard methods for singularly perturbed problems possessing oscillatory/layer solutions ☆

Jean M.-S. Lubuma [a], Kailash C. Patidar [b,*,1]

[a] Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 0002, South Africa

[b] Department of Mathematics and Applied Mathematics, University of the Western Cape, Private Bag X17, Bellville 7535, South Africa

Abstract

We construct and analyze non-standard finite difference methods for a class of singularly perturbed differential equations. The class consists of two types of problems: (i) those having solutions with layer behavior and (ii) those having solutions with oscillatory behavior. Since no fitted mesh method can be designed for the latter type of problems, other special treatment is necessary, which is one of the aims being attained in this paper. The main idea behind the construction of our method is motivated by the modeling rules for non-standard finite difference methods, developed by Mickens. These rules allow one to incorporate the essential physical properties of the differential equations in the numerical schemes so that they provide reliable numerical results. Note that the usual ways of constructing the fitted operator methods need the fitting factor to be incorporated in the standard finite difference scheme and then it is derived by requiring that the scheme be uniformly convergent. The method that we present in this paper is fairly simple as compared to the other approaches. Several numerical examples are given to support the predicted theory.

Keywords: Singular perturbations; Boundary value problems; Ordinary differential equations; Dispersivity; Dissipativity; Finite difference methods

1. Introduction

We consider the singularly perturbed two-point boundary value problem

$$
\begin{aligned}
&Ly \equiv \varepsilon y'' + a(x)y' + b(x)y = f(x) \quad \text{on } (0,1),\\
&y(0) = \alpha_0, \quad y(1) = \alpha_1,
\end{aligned}
\tag{1.1}
$$

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* Corresponding author.

E-mail addresses: jean.lubuma@up.ac.za (J.M.-S. Lubuma), kpatidar@uwc.ac.za (K.C. Patidar).

[1] Part of the research of this author was carried out while he was a post-doctoral fellow at the University of Pretoria.

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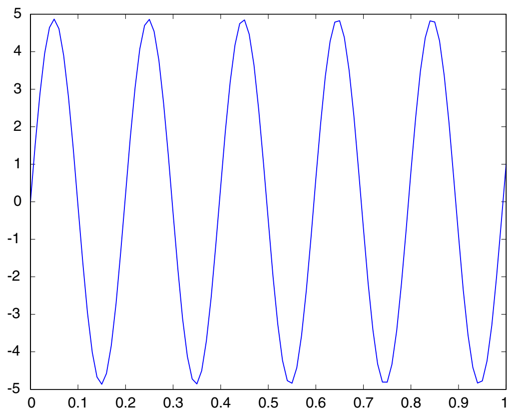
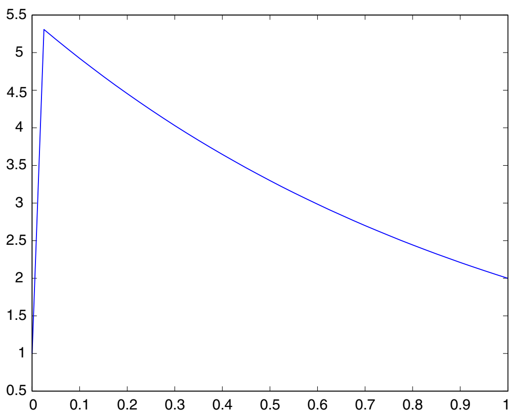
where α_0, α_1 are given constants and ε is a small positive parameter. Further, $f(x)$, $a(x)$ and $b(x)$ are sufficiently smooth functions satisfying the conditions

$$a(x) \geqslant a > 0 \quad \text{and} \quad b(x) \geqslant b > 0.$$

Problems like (1.1), in which a small parameter is multiplied to the highest derivative term, arise in various fields of science and engineering, for instance fluid mechanics, fluid dynamics, elasticity, quantum mechanics, chemical reactor theory, hydrodynamics, etc. The main concern with such problems is the rapid change of the solution in one or more narrow "layer region(s)".

The specific form of the rapid change of the solution of (1.1) depends on whether the function $a(x)$ is zero or not. The case $a(x) \equiv 0$ leads to the so-called dispersive problem whereby the solution experiences global phenomenon of rapidly oscillating throughout the entire interval $[0, 1]$. Otherwise, the problem is dissipative in the sense that the rapidly varying component of the solution decays away from a localized breakdown or discontinuity point in the layer region as $\varepsilon \to 0$. These phenomena are illustrated in the figures below. One can see that the dissipativity is indeed a local phenomenon (see bottom figure) whereas the dispersivity is a global phenomenon (see top figure):





These two figures represent the solutions of the problems

$$\varepsilon y'' + y = 0, \quad y(0) = 0, \quad y(1) = 1$$

and