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Polynomial and nonpolynomial spline approaches to the numerical solution of second order boundary value problems

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Abstract

In this paper, quadratic and cubic polynomial and nonpolynomial spline functions based methods are presented to find approximate solutions to second order boundary value problems. Using these spline functions we drive a few consistency relations which to be used for computing approximations to the solution for second order boundary value problems. The present approaches have less computational cost. Convergence analysis of these methods is discussed. Two numerical examples are included to illustrate the practical usefulness of the proposed methods.

Keywords: Quadratic and cubic polynomial splines; Nonpolynomial spline; Two-point boundary value problem; Neumann boundary conditions

1. Introduction

Many problems in science and technology are formulated mathematically in boundary value problems for second order differential equations as in heat transfer and deflection in cables.We shall consider a numerical solution of the following linear second order two-point boundary value problem, see [6]

$$y^{(2)} + f(x)y = g(x), \quad x \in [a,b], \tag{1.1}$$

subjected to Neumann boundary conditions:

$$y^{(1)}(a) - A_1 = y^{(1)}(b) - A_2 = 0, \tag{1.2}$$

where A_i, $i = 1,2$ are finite real constants. The functions $f(x)$ and $g(x)$ are continuous on the interval $[a,b]$. The analytical solution of (1.1) subjected to (1.2) can not be obtained for arbitrary choices of $f(x)$ and $g(x)$.

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The numerical analysis literature contains little on the solution of second order two-point boundary value problems (1.1) subjected to Neumann boundary conditions (1.2). Albasiny and Raghavarao [1,4] solved linear second order two-point boundary problem (1.1) subjected to Dirichlet boundary conditions using cubic polynomial spline. Blue [5] solved this problem using quintic polynomial spline. While, Caglar et al. [3] solved this problem using cubic B-spline. Arshad Khan solved it using parametric cubic spline. Zahra [6], also solved this problem using quadratic polynomial spline at midknots.

In this paper, we use both polynomial and nonpolynomial spline functions to develop numerical methods for obtaining smooth approximations for the solution of the problem (1.1) subjected to Neumann boundary conditions (1.2).

2. Derivation of the methods

We introduce a finite set of grid points x_i by dividing the interval $[a,b]$ into n equal parts.

$$\begin{aligned} &x_i = a + ih, \quad i = 0,1,\ldots,n, \\ &x_0 = a, \quad x_n = b \quad \text{and} \quad h = \frac{b-a}{n}. \end{aligned} \tag{2.1}$$

Let $y(x)$ be the exact solution of the system (1.1) and S_i be an approximation to $y_i = y(x_i)$ obtained by the spline function $Q_i(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}).

2.1. Polynomial spline forms

2.1.1. Quadratic polynomial spline

Let us write the quadratic polynomial spline $Q_i(x)$ in the form

$$Q_i(x) = a_i(x-x_i)^2 + b_i(x-x_i) + c_i, \quad i = 0,1,\ldots,n-1, \tag{2.2}$$

where a_i, b_i and c_i are constants. Thus, the quadratic spline is now defined by the relations

$$\begin{aligned} &\text{(i) } S(x) = Q_i(x), \quad x \in [x_i, x_{i+1}], \ i = 0,1,\ldots,n-1 \text{ and} \\ &\text{(ii) } S(x) \in C^1[a,b]. \end{aligned} \tag{2.3}$$

The three coefficients in (2.2) need to be obtained in terms of $S_{i+1/2}$, D_i and $F_{i+1/2}$ where

$$\begin{aligned} &\text{(i) } Q_i(x_{i+1/2}) = S_{i+1/2}, \\ &\text{(ii) } Q_i^{(1)}(x_i) = D_i, \\ &\text{(iii) } Q_i^{(2)}(x_{i+1/2}) = F_{i+1/2}. \end{aligned} \tag{2.4}$$

We obtain via a straightforward calculation

$$a_i = \frac{1}{2}F_{i+1/2}, \quad b_i = D_i, \quad c_i = S_{i+1/2} - \frac{h^2}{8}F_{i+1/2} - \frac{h}{2}D_i. \tag{2.5}$$

Now from the continuity conditions (ii) in (2.3), that is the continuity of quadratic spline $S(x)$ and its first derivative at the point (x_i, S_i), where the two quadratics $Q_{i-1}(x)$ and $Q_i(x)$ join. We have

$$Q_{i-1}^{(m)}(x) = Q_i^{(m)}(x), \quad m = 0,1.$$

Upon on using Eqs. (2.2) and (2.5) yield the relations

$$h(D_i + D_{i-1}) = 2(S_{i+1/2} - S_{i-1/2}) - \frac{h^2}{4}(F_{i+1/2} + 3F_{i-1/2}), \tag{2.6}$$

$$h(D_i - D_{i-1}) = h^2 F_{i-1/2}. \tag{2.7}$$

Adding Eqs. (2.6) and (2.7), we get

$$hD_i = (S_{i+1/2} - S_{i-1/2}) - \frac{h^2}{8}(F_{i+1/2} - F_{i-1/2}). \tag{2.8}$$