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The generalized invariance principle for dynamic equations on time scales

Wenyong Zhong, Wei Lin *, Jiong Ruan

Research Center and Key Laboratory of Mathematics for Nonlinear Sciences, School of Mathematical Sciences, Fudan University, Shanghai 200433, PR China

Abstract

The study of long-term behavior of a solution generated by dynamics on time scales always gains a great amount of attentions both from Engineers and from mathematicians. In the paper, a generalized invariance principle, which not only includes the case described by the conventional invariance principle but also involves the case where the sign of the Delta derivative of the Lyapunov function along with the solution is not determined, is established. This consequently allows us to verify the stability and boundedness of the solution generated by a much wider class of dynamic equations on time scales by utilizing this generalized principle.

Keywords: Time scales; Dynamic equations; Invariance principle; Lyapunov function

1. Introduction

In 1960s, LaSalle, inspired by the theories of the classical Lyapunov method and of the Birkhoff limit set, established the well-known invariance principle [1], which has become one of the most significant criteria to investigate the long-term behavior of a solution generated by ordinary differential equations. From then on, this principle has been consecutively extended to non-autonomous differential equations [2], to functional differential equations [3], and even to discrete dynamical systems [4]. In particular, with the development of the study of dynamic equations on time scales, the invariance principle was further extended to these general dynamic equations (see [5–7]). It could be found in the above-mentioned literature that the derivative (or the delta derivative) of the Lyapunov function for the conventional dynamic system (or for the dynamic equations on time scales) is always required to be negatively semi-definite. This requirement, however, was weakened when a more general invariance principle was proposed by Rodrigues et al. [8,9]. As a matter of fact, this general invariance principle allows the case that the derivative of Lyapunov function along with the autonomous ordinary differential equations is not necessarily semi-negative but may be positive on some bounded set,

* Corresponding author.
E-mail address: wlin@fudan.edu.cn (W. Lin).

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which provides a useful tools not only to investigate the asymptotic behavior of the corresponding solution but also to estimate its eventual boundary. Motivated by these results, we then turn to generalize the approaches to the dynamic equations:

$$\boldsymbol{x}^{\Delta} = \boldsymbol{f}(\boldsymbol{x}), \quad t \geqslant t_0, \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, t belongs to the so-called time scale $\mathbb{T}$ (an arbitrary nonempty closed subset of $\mathbb{R}$), $t_0 \in \mathbb{T}$, and $\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ within $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. Indeed, we not only establish the generalized invariance principle for the global case but also for the local case in the paper. In addition, $\boldsymbol{f}$ in Eq. (1) is throughout assumed to be continuous so that the existence and uniqueness of solutions to Eq. (1) subject to $x(t_0) = x_0$ as well as their dependence on initial values are guaranteed [10,11].

The rest of paper is organized as follows. In Section 2, some necessary definitions of dynamic equations on time scales as well as properties of limit set are preliminarily introduced. In Section 3, both the generalized global invariance principle and the local principle are, respectively, established. Apart from this, the generalized invariance principle for some particular cases of time scales, such as discrete dynamical system, is also clarified in this section. In Section 4, concrete examples are further provided to illustrate the possible application of the generalized invariance principle. Finally, the paper is closed by some conclusions and discussions in Section 5.

2. Preliminaries

2.1. Time scales

In order to make the paper self-contained, we first introduce necessary definitions and lemma on time scales (for details, refer to [11]). A time scale $\mathbb{T}$ is any nonempty closed subset of real numbers $\mathbb{R}$ with order and topological structure defined in a canonical way.

Definition 2.1. For $t \in \mathbb{T}$ defined the forward jump operator $\sigma : \mathbb{T} \mapsto \mathbb{T}$ and backward jump operator $\rho : \mathbb{T} \mapsto \mathbb{T}$, respectively by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

The graininess operator $\mu : \mathbb{T} \mapsto [0, \infty)$ is defined as $\mu(t) = \sigma(t) - t$.

Throughout the paper, the closed interval in $\mathbb{T}$ is defined as $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. Accordingly, open interval and half-open interval are defined.

Definition 2.2 (*Delta derivative*). Assume that $f : \mathbb{T} \mapsto \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Define $f^{\Delta}(t)$ to be the number (provided that it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leqslant \epsilon |\sigma(t) - s|, \quad \text{for all } s \in U.$$

Then the function $f^{\Delta}(t)$ is said to be the delta derivative of f at t.

Definition 2.3. A function $f : \mathbb{T} \mapsto \mathbb{R}$ is right dense continuous (rd-continuous) provided that it is continuous at all right dense points of $\mathbb{T}$ and its left side limits exists (finite) at left dense points of $\mathbb{T}$. The set of all right dense continuous functions on $\mathbb{T}$ is denoted by $C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T})$.

Definition 2.4 (*Delta integral*). Let $f : \mathbb{T} \mapsto \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. if there exists a function $F : \mathbb{T} \mapsto \mathbb{R}$. such that $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$, Then F is a delta derivative of f. In this case the integral is given by the formula

$$\int_a^b f(\tau) \Delta\tau = F(b) - F(a), \quad \text{for all } a, b \in \mathbb{T}.$$