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Unconditional stability of parallel difference schemes with second order accuracy for parabolic equation ☆

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Abstract

In this paper we investigate the parallel difference schemes of parabolic equation, in particular, two kinds of difference schemes with intrinsic parallelism are constructed. Firstly we combine the values of previous two time levels at the interface points to get the (Dirichlet) boundary condition for the sub-domain problems. Then the values in the sub-domains are calculated by fully implicit scheme. And then finally the values at the interface points are computed by fully implicit scheme. The unconditional stability of these schemes is proved, and the convergence rate of second order is also obtained. Numerical results are presented to examine the accuracy, stability and parallelism of the parallel schemes.

Keywords: Parabolic equation; Parallel difference; Unconditional stability; Convergence; Second-order accuracy

1. Introduction

There is rich literature on the parallel difference schemes for the parabolic equation (see [1–10]). Explicit schemes are often naturally parallel and also easy to implement, but they usually require small time steps because of stability constraints. Implicit schemes are necessary for finding steady state solution or computing slowly unsteady problems where one needs to march with large time steps. However, implicit schemes are not inherently parallel.

The alternating schemes in [1–5] use the explicit scheme and implicit scheme alternately in the time and space direction, which can implement the parallel computation and are unconditionally stable, i.e., for *any* positive constant C, when $\lambda \leqslant C$, the scheme is stable. For the heat equation $u_t = u_{xx}$ the classic explicit scheme is not unconditional stable since C cannot be taken larger than $\frac{1}{2}$, however the classic implicit scheme and some alternating schemes are unconditional stable. Note that these alternating schemes can not be implemented directly by making use of the original sequential codes.

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Domain decomposition is a powerful tool for devising parallel PDE methods. Much of the work on domain decomposition has been directed at elliptic equation (see [11]). In this paper, we present a finite difference method which utilizes domain decomposition to allow us to divide the work of solving the heat equation. The method differs from the methods mentioned in [11] that it is noniterative. It can be used to divide the global problem into smaller sub-domain problems, which can be solved in parallel. The major difficulties with such procedures involve defining values on the sub-domain boundaries and piecing the solutions together into a reasonable approximation to the true solution. Once the interface values are available, the global problem is fully decoupled and thus be computed in parallel. A domain decomposition scheme was proposed in [6], where instead of using the same spacing h as for the interior points where the implicit scheme is applied, a larger spacing H_D is used at each interface points where the explicit scheme is applied. Due to stability and accuracy requirement, the method does not lead to satisfactory parallel efficiency, although the method can be implemented with little extra effort using the original sequential codes. there also have some other schemes with domain decomposition in [7,8]. These schemes are generally of second order global accuracy in space, i.e., $O(\Delta t + h^2)$, but all of them are conditionally stable.

The unconditional stable scheme is desirable in solving some application problems. Some unconditionally stable schemes were proposed in [9], which firstly take the values of previous time step as the boundary condition, then solve the sub-domain problems by implicit scheme in parallel, and then finally update the interface values by implicit scheme. These schemes can be easily implemented using the original sequential codes, but they are only of one order global accuracy in space. In order to improve the global accuracy, the parallel iterative difference schemes based on interface correction were proposed in [10]. But it can not assure second order global accuracy in theory.

Up until now the available parallel difference schemes at least have one of the following defects: (1) they are conditionally stable (e.g. [6–8]), i.e., there exists a positive constant $C < +\infty$ such that the schemes are unstable when $\lambda > C$; (2) the accuracy of these parallel schemes is lower than fully implicit scheme (e.g. [1–5,9]); (3) the design of algorithm is complex (e.g. [1–5,10]), and therefore the scheme can not be implemented with little extra effort with the original sequential codes.

In this paper, we propose two kinds of parallel difference scheme with intrinsic parallelism. The resulting schemes are of second order global accuracy in space and unconditionally stable. The principle of the method lies mainly in the following steps. First we get the inner boundary condition by combining the values of previous two time levels at the interface points, then compute the values in the sub-domain by fully implicit scheme, and then finally update the interface values by fully implicit scheme. When the interface values are obtained by the linear combination, the global problem is fully decoupled and can thus be computed in parallel. No iterations between sub-domains are necessary. Hence, our method should be nearly optimal. The calculations of updating the interface values are explicit, since the values adjacent to the interface points have been obtained. The time needed to update the interface values is very small relative to the time needed to solve sub-domain problems. There only need to pass information between neighboring sub-domains and no global communication is necessary. Thus, the parallel algorithm is fully scalable. Furthermore, the method is easy to incorporate into existing implicit codes.

The rest of this paper is organized as follows. In the next section, we propose two kinds of parallel difference scheme for a one-space-dimensional problem. In Section 3, we prove the stable and convergence result for these schemes. The resulting schemes are unconditionally stable and convergence in the sense of discrete H^1 norm, and of second order accuracy in space, i.e., $O(\Delta t + h^2)$. In Sections 4 and 5, we extend the one-dimensional results to two space dimensions, and prove the stability and convergence results. In Section 6, we examine numerically the accuracy, stability and parallelism of the scheme on certain test problems. The numerical results verify the theoretical results. Moreover it is shown that the super-linear speedup is achieved. In Section 7, we give some generalizations and conclusions.

2. Construction of one-dimensional schemes

2.1. Problem and notation

Let $U(x, t)$ be the solution of the following model problem with the initial and boundary conditions.