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ScienceDirect
Applied Mathematics and Computation 186 (2007) 1326–1333
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Topological mappings between graphs, trees and generalized trees

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Abstract

We present novel topological mappings between graphs, trees and generalized trees that means between structured objects with different properties. The two major contributions of this paper are, first, to clarify the relation between graphs, trees and generalized trees, a graph class recently introduced. Second, these transformations provide a unique opportunity to transform structured objects into a representation that might be beneficial for a processing, e.g., by machine learning techniques for graph classification.

Keywords: Applied graph theory; Graph transformation; Hierarchical graphs

1. Introduction

The number of different graph classes known so far is large [2]. For example, undirected or directed, weighted or unweighted, labeled or unlabeled graphs are know since a long time [2,3,13] and frequently used to represent physical [9], biological or sociological objects [1,10,16,17], e.g., a crystal structure, a protein structure [11,12] or the acquaintance network between a group of people, to enable a mathematical treatment. In this paper we investigate the relation between three important graph classes: graphs, trees and generalized trees. The latter class was recently introduced by [14,4] and, hence, is a new graph class largely unexplored so far. One important result of this study is that we are able to clarify the relation between a graph and a generalized tree by introducing a transformation that maps a graph on a generalized tree and vice versa. We will see, that a transformation between an unhierarchical graph and a hierarchical generalized tress can only be obtained if one introduces a multi-level function which assigns each node of the graph a level value introducing a hierarchy. In addition to general transformations between graphs, trees and generalized trees we introduce a special transformations which uses the Dijkstra distance between nodes to define the multi-level

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doi:10.1016/j.amc.2006.07.162

function. These transformations, we call natural transformations, are especially useful for algorithmic applications, because, e.g., the transformation form a graph to a generalized tree can be easily implemented.

This paper is organized as follows: In Section 2 mathematical preliminaries are defined to enable a better understanding for the following sections. In Section 3 we introduce a general transformation between a graph and a generalized tree and vice versa. Section 4 presents a special case of the general transformation of the previous section between graphs, generalized trees and also trees. Some practical examples are provided to visualize the introduced procedure. The article finishes in Section 5 with conclusions.

2. Mathematical preliminaries

Given a directed or undirected, weighted or unweighted, labeled or unlabeled graph G with vertex set V, edge set E and $N = |V|$ nodes. The graph G does not include any hierarchy between the nodes for this reason we say without loss of generalization that graph G consists of only one hierarchy level. More formally, each node of the vertex set V is assigned to the same level we call $0 \in L$. Here L is the level set containing identifier of all levels. The graph class *generalized tree* was introduced in [14,4]. However, here we give an alternative definition emphasizing that nodes that belong to the same level are not ordered.

Definition 1 (*Generalized tree*). A generalized tress GT_i is defined by a vertex set V, an edge set E, a level set L and a multi-level function $\mathscr{L}_i$. The vertex and edge set define the connectivity and the level set and the multi-level function induces a hierarchy between the nodes of GT_i. The index $i \in V$ indicates the root.

The multi-level function is defined as follows.

Definition 2 (*Multi-level function*). The surjective function $\mathscr{L}_i : V \setminus \{i\} \rightarrow L$ is called multi-level function.

The multi-level function $\mathscr{L}_i$ assigns to all nodes except i an element $l \in L$ that corresponds to the level it will be assigned. From these definitions it is immediately clear that a generalized tree is similar to a graph but additionally equipped with a level set L and a multi-level function $\mathscr{L}_i$ that introduce a node grouping corresponding to the introduction of a hierarchy between nodes and sets thereof. In Fig. 1 we visualize a graph G and a generalized tree GT_3. For reasons of simplicity we show an undirected, unweighted generalized tree, however,

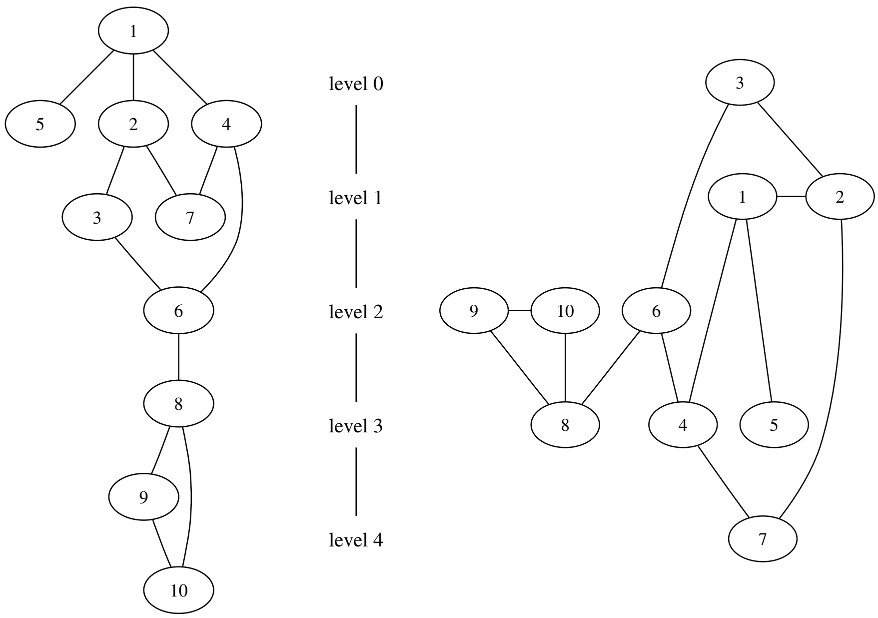

Fig. 1. Left: An undirected, unweighted and labeled graph G. Right: An undirected, unweighted and labeled generalized tree GT_3 rooted at node 3. To emphasize the hierarchical nature of GT_3 we included a panel indicating the level the nodes are assigned corresponding to the hierarchy.