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An iterative method for symmetric solutions and optimal approximation solution of the system of matrix equations $A_1XB_1 = C_1,\ A_2XB_2 = C_2$ ☆

Ya-Xin Peng [a,b,*], Xi-Yan Hu [a], Lei Zhang [a]

[a] College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China
[b] School of Mathematical and Statistics, Central South University, Changsha, Hunan 410083, PR China

Abstract

The symmetric solutions of the system of matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ are too difficult to be obtained by applying matrices decomposition. In this paper, an iterative method is applied to solve this problem. With it, the solvability of this system of matrix equations can be determined automatically, when this system of matrix equations is consistent, its solution can be obtained within finite iterative steps, and its least-norm solution can be obtained by choosing a special kind of initial iterative matrix, furthermore, its optimal approximation solution to a given matrix can be derived by finding the least-norm symmetric solution of a new system of matrix equations $A_1\tilde{X}B_1 = \tilde{C}_1,\ A_2\tilde{X}B_2 = \tilde{C}_2$. Finally, numerical examples are given.

Keywords: Iterative method; Symmetric solution; Least-norm solution; Optimal approximation

1. Introduction

Let $R^{m\times n}$ denote the set of all $m \times n$ real matrices, $R^m = R^{m\times 1}$, $SR^{n\times n}$ denote the set of all symmetric matrices in $R^{n\times n}$. For a matrix $A \in R^{m\times n}$, $\|A\|$ represents its Frobenius norm, $R(A)$ represents its column space, vec$(\cdot)$ represents the vec operator, i.e. $\mathrm{vec}(A) = \left(a_1^{\mathrm{T}}, a_2^{\mathrm{T}}, \ldots, a_n^{\mathrm{T}}\right)^{\mathrm{T}}$ for the matrix $A = (a_1, a_2, \ldots, a_n) \in R^{m\times n}$, $a_i \in R^m$, $i = 1, 2, \ldots, n$. $A \otimes B$ stands for the Kronecker product of matrices A and B.

In this paper, we consider the following problems.

Problem I. Given $A_1 \in R^{m_1\times n}$, $B_1 \in R^{n\times p_1}$, $C_1 \in R^{m_1\times p_1}$, $A_2 \in R^{m_2\times n}$, $B_2 \in R^{n\times p_2}$, $C_2 \in R^{m_2\times p_2}$, find $X \in SR^{n\times n}$, such that

$$\begin{cases} A_1XB_1 = C_1, \\ A_2XB_2 = C_2. \end{cases} \tag{1}$$

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* Corresponding author. Address: College of Mathematics and Econometrics, Hunan University, Changsha 410082, PR China.
E-mail address: Yaxinpeng@hnu.cn (Y.-X. Peng).

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Problem II. When Problem I is consistent, let S_E denote the set of solutions of Problem I, for given $X_0 \in R^{n\times n}$, find $\hat{X} \in S_E$, such that

$$\|\hat{X} - X_0\| = \min_{X\in S_E} \|X - X_0\|. \tag{2}$$

In fact, Problem II is to find the optimal approximation solution to the given matrix X_0.

Research on solving systems of linear matrix equations has been actively ongoing for past years. For e.g., Mitra [1,2] has provided conditions for the existence of a solution and a representation of the general common solution to the matrix equations $AX = C$, $XB = D$ and the matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$. Also, Navarra et al. [3] studied a representation of the general common solution to the matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$, Bhimasankaram [4] considered the linear matrix equations $AX = B$, $CX = D$ and $EXF = G$, van der Woude [5] obtained the existence of a common solution X to the matrix equations $A_iXB_j = C_{ij}$, $(i,j) \in \Gamma$. Till now, the problem to obtain symmetric solutions of the system of matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ has not been solved.

In this paper, an iterative method is constructed to solve the system of matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ over symmetric X. With it, the solvability of the system of matrix equations can be determined automatically, when the system of matrix equations is consistent, its symmetric solution can be obtained within finite iterative steps, and its least-norm solution can be obtained by choosing a suitable initial iterative matrix, furthermore, its optimal approximation solution to a given matrix can be derived by finding the least-norm symmetric solution of a new system of matrix equation $A_1\tilde{X}B_1 = \tilde{C}_1,\ A_2\tilde{X}B_2 = \tilde{C}_2$.

2. An iterative method for solving Problem I

The conjugate gradients method is an efficient method to solve linear systems $Ax = b$, where A is symmetric and positive definite, $x \in R^n$ (see [6–8]). For nonsymmetric matrix A, we can use general conjugate gradients methods in [9–11]. Similarly, we construct an iterative method to obtain the symmetric solutions of the system of matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$.

Step 1: Choose an arbitrary matrix $X_1 \in SR^{n\times n}$, compute

$$R_1 = \begin{pmatrix} C_1 - A_1X_1B_1 & 0 \\ 0 & C_2 - A_2X_1B_2 \end{pmatrix},$$
$$P_1 = A_1^{\mathrm{T}}(C_1 - A_1X_1B_1)B_1^{\mathrm{T}} + A_2^{\mathrm{T}}(C_2 - A_2X_1B_2)B_2^{\mathrm{T}},$$
$$Q_1 = \frac{1}{2}(P_1 + P_1^{\mathrm{T}}).$$
$$k := 1.$$

Step 2: Compute

$$X_{k+1} = X_k + \frac{\|R_k\|^2}{\|Q_k\|^2} Q_k.$$

Step 3: Compute

$$R_{k+1} = \begin{pmatrix} C_1 - A_1X_{k+1}B_1 & 0 \\ 0 & C_2 - A_2X_{k+1}B_2 \end{pmatrix},$$
$$P_{k+1} = A_1^{\mathrm{T}}(C_1 - A_1X_{k+1}B_1)B_1^{\mathrm{T}} + A_2^{\mathrm{T}}(C_2 - A_2X_{k+1}B_2)B_2^{\mathrm{T}},$$
$$Q_{k+1} = \frac{1}{2}(P_{k+1} + P_{k+1}^{\mathrm{T}}) - \frac{\mathrm{tr}(P_{k+1}Q_k)}{\|Q_k\|^2} Q_k.$$

If $R_{k+1} = 0$, or $R_{k+1} \neq 0$, $Q_{k+1} = 0$, stop, otherwise, let $k := k + 1$, go to Step 2.

Obviously, we know that $Q_i \in SR^{n\times n}$, $X_i \in SR^{n\times n}$, where $i = 1, 2, \ldots$