Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 182 (2006) 119–139
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

ε-Uniformly convergent fitted methods for the numerical solution of the problems arising from singularly perturbed general DDEs

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Abstract

We consider some problems arising from singularly perturbed general differential difference equations. First we construct (in a new way) and analyze a "fitted operator finite difference method (FOFDM)" which is first order ε-uniformly convergent. With the aim of having just one function evaluation at each step, attempts have been made to derive a higher order method via Shishkin mesh to which we refer as the "fitted mesh finite difference method (FMFDM)". This FMFDM is a direct method and ε-uniformly convergent with the nodal error as $\mathcal{O}(n^{-2}\ln^2 n)$ which is an improvement over the existing direct methods (i.e., those which do not use any acceleration of convergence techniques, e.g., Richardson's extrapolation or defect correction, etc.) for such problems on a mesh of Shishkin type that lead the error as $\mathcal{O}(n^{-1}\ln n)$ where n denotes the total number of sub-intervals of $[0,1]$. Comparative numerical results are presented in support of the theory.

Keywords: Differential difference equations; Singular perturbations; Boundary value problems; Fitted operator methods; Fitted mesh methods; Shishkin mesh

1. Introduction

Consider the singularly perturbed differential difference equation (SPDDE):

$$\varepsilon y''(x) + a(x)y'(x) + \alpha(x)y(x-\delta) + \zeta(x)y(x) + \beta(x)y(x+\eta) = f(x) \quad \text{on } \Omega = (0,1), \tag{1.1}$$

under the interval and boundary conditions

$$\begin{aligned} y(x) &= \phi(x) \quad \text{on } -\delta \leqslant x \leqslant 0, \\ y(x) &= \gamma(x) \quad \text{on } 1 \leqslant x \leqslant 1+\eta, \end{aligned} \tag{1.2}$$

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.03.043

where $a(x)$, $\alpha(x)$, $\zeta(x)$, $\beta(x)$, $f(x)$, $\phi(x)$ and $\gamma(x)$ are sufficiently smooth functions, $0 < \varepsilon \ll 1$ is the singular perturbation parameter and $0 < \delta = o(\varepsilon)$, $0 < \eta = o(\varepsilon)$ are the delay and the advance parameters (sometimes referred to as 'negative shift' and 'positive shift', respectively, as in [17,18]). Here it is worthwhile to mention that some of the analytical results (when $\delta = \eta = 0$) being proved in this paper, supplements the existing theory for FOFDMs and FMFDMs for singular perturbation problems (SPPs).

The solution of (1.1) and (1.2) exhibits layer at the left or right end of the interval depending on whether $a(x) - \alpha(x)\delta + \beta(x)\eta > 0$ or <0 on $[0,1]$. In case, if $a(x) \equiv 0$ then one may have oscillatory solutions or two layers (one at each end) depending upon the cases whether $\alpha(x) + \zeta(x) + \beta(x)$ is positive or negative, both of which are considered in [29].

The arguments for small delay problems are found throughout the literature on epidemics and population where these small shifts play an important role in the modeling of various real life phenomena [16]. For example, in the mathematical model for the determination of the expected first-exit time in the generation of action potential in nerve cells by random synaptic inputs in dendrites, the shifts are due to the jumps in the potential membrane which are very small. Numerous researchers have observed that often the shifts are very small and affect the solution significantly. By considering several numerical examples, Lange and Miura [18] have shown the effect of very small shifts (of the order of ε) on the solution and pointed out that they drastically affect the solution and therefore cannot be neglected. Further studies of the effect of δ and η on the layer behavior of the solution has been carried out by Kadalbajoo and Sharma [15]. For the occurrence and further motivation for solving such problems, the readers may refer to the introduction section in [14] and the references therein. Some of the other relevant references are [2,17,18,20,36,37].

In this paper, first we present some analytical results for the case when there is one boundary layer at the left end of the interval. The case when the layer occurs at right end, can be analyzed similarly. However, we do present some numerical results for the later case. On the other hand, we construct two types of methods: Fitted operator finite difference method (FOFDM) and fitted mesh finite difference method (FMFDM).

The first aim of this paper is to show a new way of designing these FOFDMs. Like the other existing ways to get the similar schemes, its simplicity is explained by incorporating the method on some difficult problems than those considered earlier while still gaining ε-uniform numerical results. The next step is about how to improve the order of convergence of the method? It is possible with the same class of methods but one requires more function evaluations at each step. See, e.g., [27] in which considering three evaluations of the function at each step, a fourth order method has been presented (here function means the right-hand side function $f(x)$). However, there are no other ways (to the best of our knowledge) to get a ε-uniformly convergent method (in the sense of Definition 1.1) of high order with just one function evaluation per step via any FOFDM. To this end, it is natural to ask whether there are other class of methods which can give higher order convergence with only one function evaluation per step? The answer is yes. These are the carefully designed fitted mesh methods which use the same number of grid points (of course distributed differently). The FMFDM thus obtained is applied to the class of problems whose solution has a layer at left end. Since with some appropriate changes in the transition parameters of the mesh, the method is analogously applicable to the problems whose solution has a layer at right end, we omit that part. We also provide some ε-uniform numerical results with this FMFDM for a problem when $a(x) \equiv 0$ in (1.1).

As far as the SPDDEs are concerned, the idea of the FOFDM considered in this paper has successfully been implemented for the first time by the authors in [30] where they considered the problems having small delay. The FOFDM presented in this paper is obtained via so-called nonstandard finite difference method (NSFDM). The method is nonstandard due to the fact that the classical denominator h or h^2 of the discrete first or second order derivative in the standard finite difference scheme is replaced by a nonnegative function ϕ such that

$$\phi(z) = z + O(z^2) \text{ or } \phi(z) = z^2 + O(z^3) \quad \text{as } 0 < z \to 0. \tag{1.3}$$

This denominator function replicates a number of significant properties of the governing differential equation. Interested readers may refer to a comprehensive list of works on NSFDMs (till early 2005) in a recent survey article [28] by the second author of this paper. On the other hand, a special section is devoted in the book chapter [21] towards the definition of these NSFDMs.