Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 182 (2006) 200–209
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Discrete distributions from moment generating function

P.L. Novi Inverardi, A. Tagliani *

Department of Computer and Management Sciences, University of Trento, via Inama, 1 – 38100 Trento, Italy

Abstract

The recovering of positive discrete distributions from their moment generating function (*mgf*) is considered. From *mgf* and some integer moments, proper fractional moments are obtained. The latter represent the available information of the distribution. Then maximum entropy machinery is invoked to find the approximate distribution. It is proved that the approximant converges in entropy, in information divergence and then in total variation, so that accurate expected values may be obtained. Some numerical experiments are illustrated.

Keywords: Entropy; Fractional moments; Hankel matrix; Integer moments; Maximum entropy; Moment generating function

1. Introduction

In Applied Probability and Statistics, the moment generating function (*mgf*) is usually considered as a valuable tool for calculating moments or for identifying the distribution of a sum of independent random variables. When the underlying distribution is absolutely continuous, the corresponding density is recovered by *mgf* inversion, lending the plethora of Laplace transform inversion techniques. On the contrary, if the underlying distribution is discrete, inversion techniques seem lacking. On the other hand, discrete distributions have many important applications in the study of queueing systems [8] and in risk theory. In this paper, we propose a probability mass function (*pmf*) reconstruction procedure from its *mgf* based on the Maximum Entropy approach passing through fractional moments.

It is a well-known fact that, given M pieces of information $I_1, I_2, \ldots, I_M$, in terms of expected values, the Maximum Entropy method [5] allows us to recover the Shannon-entropy maximizing probability mass function (*pmf*) $p^{(M)}$, which is the most consistent and coherent with (and only) the information contained in the M quantities $I_1, I_2, \ldots, I_M$. However, in order that the reconstructed *pmf* be operative, the quantities $I_1, I_2, \ldots, I_M$ must characterize the underlying distribution. This requirement is not a negligible aspect of the procedure and the most popular choice of $I_1, I_2, \ldots, I_M$ is consisted in the integer moments which, in many (although not in all) cases, characterize (= guarantee the existence of a unique distribution having these moments) the distribution. Recently, Novi Inverardi and Tagliani [11] proposed to involve fractional moments in density

* Corresponding author.
E-mail address: aldo.tagliani@economia.unitn.it (A. Tagliani).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.03.048

recovering on the basis of the following motivations: first, there is a result of Lin [10] which supports the characterization of a distribution through its fractional moments and second, the Maximum Entropy *pmf* $p^{(M)}$ recovered involving fractional moments converges in entropy to the true *pmf* p. This last result means that if we are interested in approximating some characteristic constants of a discrete distribution (with respect to expected values, probabilities or other), then the equivalent counterparts evaluated on $p^{(M)}$ are as close as we like to the true values, and the closeness depends on the (increasing) value of M.

For recovering a distribution (continuous or discrete) via Maximum Entropy setup, fractional moments are definitely better than integer moments, as shown in Novi Inverardi and Tagliani [11]. But they are not always easy to evaluate. Traditionally, the *mgf* of a random variable X is used to generate positive integer moments of X. But it is clear that the *mgf* also contains a wealth of knowledge about arbitrary real moments and hence, on fractional moments. Taking this into account, to obtain fractional moments Cressie and Borkent [3] exploited the *mgf* and its derivatives and Klar [7], in addition to it, exploited the knowledge of a set of integer moments which can be obtained by proper integration of the *mgf* on a contour $\mathscr{C}$ of the complex plane. We will use these results to produce the building blocks, i.e., the fractional moments of our discrete distribution reconstruction procedure. Once the fractional moments are available through the usual Maximum Entropy machinery, the approximant $p^{(M)}$ of the *pmf* p will be obtained.

2. Fractional moments from *mgf*

Let $X = \{x_1, x_2, \ldots\}$ be a countable discrete positive random variable with *pmf* $p = \{p_1, p_2, \ldots\}$, $M(t)$, t complex, its *mgf*, $\{\mu_j\}_{j=1}^{\infty}$ its infinite sequence of integer moments (which characterize X). From $M(t)$, fractional moments $\mathbb{E}(X^\alpha) = \sum_i x_i^\alpha p_i$ may be obtained through proper procedures and some of them are listed below:

1. By repeated differentiation of *mgf* by hand or through a symbolic manipulation language, such as MACSYMA, MATHEMATICA or MAPLE, the fractional calculus provides [3] $\mathbb{E}(X^\alpha)$, where

$$\mathbb{E}(X^\alpha) = \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{0} (-z)^{\lambda-1} \frac{\mathrm{d}^n M(z)}{\mathrm{d}z^n} \mathrm{d}z, \quad \lambda = n - \alpha, \ \lambda \in (0,1), \ n \in \mathbb{N}. \tag{2.1}$$

   Here, $\Gamma(\cdot)$ indicates the Gamma function in (2.1) and the real values of $M(t)$ only are requested.

2. Whenever repeated differentiation is a difficult task, $\mathbb{E}(X^\alpha)$, $0 < \alpha < 1$, may be obtained through integration, according to Hoffmann-Jorgensen [4]:

$$\mathbb{E}(X^\alpha) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{1 - M(-z)}{z^{\alpha+1}} \mathrm{d}z, \quad 0 < \alpha < 1. \tag{2.2}$$

   It is an easy task to prove that (2.2) stems from (2.1) by setting $n = 1$.Recalling that $\mathbb{E}(X^\alpha) = \sum_i x_i^\alpha p_i$ is an analytic function then if α^*, $0 < \alpha^* < 1$, is an arbitrary fixed value of α, we may have $\mathbb{E}(X^{\alpha^*})$ and the first n derivatives $\frac{\mathrm{d}^j}{\mathrm{d}\alpha^j} \mathbb{E}(X^\alpha)|_{\alpha^*}$, $j = 1, \ldots, n$ as

$$\frac{\mathrm{d}^j}{\mathrm{d}\alpha^j} \left[\frac{\Gamma(1-\alpha)}{\alpha} \mathbb{E}(X^\alpha) \right] \Bigg|_{\alpha^*} = (-1)^j \int_0^{\infty} \ln^j z \frac{1 - M(-z)}{z^{\alpha^*+1}} \mathrm{d}z, \quad j = 1, \ldots, n \tag{2.3}$$

   from which, through Taylor expansion, the approximate values of $\mathbb{E}(X^\alpha)$ can be obtained as

$$\mathbb{E}(X^\alpha) \simeq \mathbb{E}^{\mathrm{appr}}(X^\alpha) = \sum_{j=0}^{n} \frac{\mathbb{E}^{(j)}(X^{\alpha^*})}{j!} (\alpha - \alpha^*)^j. \tag{2.4}$$

   $\mathbb{E}^{\mathrm{appr}}(X^\alpha)$ is defined within an interval centered in α^*, whose radius R is governed by the abscissa of convergence of $\mathbb{E}(X^\alpha)$.

3. From $M(t)$ and some integer moments, $\{\mu_j\}_{j=1}^{M-1}$, fractional moments may be obtained, according to Klar [7], as

$$\mathbb{E}(X^{r+M-1}) = (-1)^M \frac{\prod_{j=0}^{M-1}(r+j)}{\Gamma(1-r)} \int_0^{\infty} s^{-r-M} \left[M(-s) - \sum_{j=0}^{M-1} (-1)^j \frac{\mu_j s^j}{j!} \right] \mathrm{d}s \tag{2.5}$$