Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 182 (2006) 325–332
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

On a unified mixture distribution

R.K. Saxena [a], S.L. Kalla [b,*]

[a] Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur 342 004, India
[b] Department of Mathematics and Computer Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait

Abstract

In this article, a new mixture distribution associated with Fox–Wright generalized hypergeometric function has been studied, which generalizes many mixture distributions investigated earlier by many authors. Some basic functions associated with the probability density function of the mixture distribution, such as kth moments, characteristic function and factorial moments are derived. Some special cases are also pointed out.

Keywords: Mixture distribution; ω-Hyper Poisson distribution; Probability density function; Kampé de Fériet's function; ω-Confluent hypergeometric function

1. Introduction

A particular mixture distribution stems when all or some parameters of a distribution vary according to certain given probability distribution, called the mixing distribution. A well-known example is of Poisson distribution mixture with gamma mixing distribution leading to negative binomial distribution. Such distributions are applicable in problems associated with accident proneness [8] and entomological field data [8].

Ben Nakhi and Kalla [6] investigated some mixture distributions, which are obtained by mixing discrete distributions with continuous ones. In a recent paper [7] these distributions are further extended. Probability distributions associated with Special functions of mathematical physics are introduced by many authors. In this connection, the interested reader can see the works [4,16,17] for generalized gamma type, [1] for inverse Gaussian distribution, using a generalized form of Kobayashi's gamma function [19], and [2] for distribution involving confluent hypergeometric functions of two variables due to Al-Saqabi and Kalla. The present article deals with a mixture distribution which provides generalization of the distributions discussed in [6–8,13]. Section 2 gives definitions of some special functions alongwith their important properties and some basic results used in this article. Section 3 contains a new mixture distribution obtained by mixing generalized Hyper Poisson distribution, denoted by $f(x/\lambda)$, defined by Eq. (3.2) and a new generalized gamma distribution $g(\lambda)$ defined in (3.1). Kth order moments, characteristic function, factorial moments are derived for this new mixture distribution.

* Corresponding author.
E-mail addresses: kalla@mcs.sci.kuniv.edu.kw, shyamkalla@yahoo.com (S.L. Kalla).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.02.049

The results obtained for this mixture distribution in this article are of general character and include the works investigated earlier by Ghitany et al. [13], Ben Nakhi and Kalla [6,7] and Bhattacharya [8].

2. Mathematical prerequisites

The function denoted by ${}_p\Psi_q$ is called the Wright (or, more appropriately, the Fox–Wright) generalization of the hypergeometric function ${}_pF_q$, which is defined by {[11,33,34], see also [10, p. 183] and [29, p. 21]} in terms of the series as

$$
{}_p\Psi_q(z) \equiv {}_p\Psi_q\left[\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\middle| z\right] = {}_p\Psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p)\\(b_1, B_1), \ldots, (b_q, B_q)\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i + A_i k)}{\prod_{j=1}^{q}\Gamma(b_j + B_j k)} \frac{z^k}{k!}, \tag{2.1}
$$

where $A_j > 0$ $(j = 1, \ldots, p)$; $B_j > 0$ $(j = 1, \ldots, q)$, $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqslant 0$, for suitably bounded values of $|z|$. Wright [33,34], investigated its asymptotic behavior. Some properties of this function were discussed in [20]. This function is an entire function of z, as shown by Kilbas et al. [18]. In recent years, this function has gained importance due to its occurrence as the solution of certain fractional reaction and fractional diffusion problems of science and engineering. In this connection, one can refer to the work reported in the papers [14,15,20,22].

When $A_i = B_j = 1$ $\forall i,\ j \in N$, ${}_p\Psi_q(z)$ reduces to a generalized hypergeometric function, ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$, defined by [10, 4.1(1)]:

$$
{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{2.2}
$$

$$
(a_i, b_j \in C);\ \ b_j \neq 0, -1, -2, \ldots \quad (i = 1, \ldots, p;\ j = 1, \ldots, q), \quad p < q + 1 \text{ (or } p = q + 1 \text{ and } |z| < 1), \tag{2.3}
$$

where $(a)_k$ is the so-called Pochhammer's symbol defined for $a \in C$ by [10, 4.1.2]

$$
(a)_0 = 1, \quad (a)_k = a(a+1) \cdots (a+k-1) \quad (k = 1, 2 \ldots). \tag{2.4}
$$

The Laplace transform of the Fox–Wright generalized hypergeometric function has been obtained by Srivastava et al. [30, p. 944] as

$$
\int_0^{\infty} t^{\rho-1} \mathrm{e}^{-st}\, {}_p\Psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p)\\(b_1, B_1), \ldots, (b_q, B_q)\end{matrix}\middle| -zt^{\sigma}\right] \mathrm{d}t = s^{-\rho}\, {}_{p+1}\Psi_q\left[\begin{matrix}(\rho, \sigma), (a_1, A_1), \ldots, (a_p, A_p)\\(b_1, B_1), \ldots, (b_q, B_q)\end{matrix}\middle| -z/t^{\sigma}\right], \tag{2.5}
$$

where $Re(\rho) > 0$, $Re(s) > 0$, $\sigma \in \Re^+$, $\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqslant \sigma$.

The following integral gives the Laplace transform of the product of two Fox–Wright generalized hypergeometric functions in terms of the S-function, defined and studied by Srivastava and Daoust [27,28]:

$$
\begin{aligned}
&\int_0^{\infty} t^{\rho-1} e^{-st}\, {}_p\Psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p)\\(b_1, B_1), \ldots, (b_q, B_q)\end{matrix}\middle| -bt\right] {}_p\Psi_q\left[\begin{matrix}(c_1, \gamma_1), \ldots, (c_p, \gamma_p)\\(d_1, \delta_1), \ldots, (d_q, \delta_q)\end{matrix}\middle| -ct\right] \mathrm{d}t \\
&\quad = s^{-\rho} S^{1:p,p}_{0:q,q}\left(\begin{matrix}[\rho : 1, 1] : [a_p :, A_p]; [c_p : \gamma_p];\\ - : [b_q : B_q]; [d_q : \delta_q];\end{matrix}\middle| \begin{matrix}-b/s\\-c/s\end{matrix}\right),
\end{aligned} \tag{2.6}
$$

where $Re(\rho) > 0$, $Re(s) > 0$, $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqslant 0$, $1 + \sum_{j=1}^{q} \gamma_j - \sum_{j=1}^{p} \delta_j \geqslant 0, p < q$.

Eq. (2.6) can be established by expressing the Fox–Wright generalized hypergeometric functions in terms of their equivalent series, interchanging the order of integration and summations, which is permissible under the conditions stated alongwith the result, applying the Laplace integral and interpreting the result thus obtained with the help of (2.7).

Following, Srivastava and Daoust [27], the S-function occurring in the above equation is defined by means of the double series in the form: