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Peakons, kinks, compactons and solitary patterns solutions for a family of Camassa–Holm equations by using new hyperbolic schemes

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Abstract

In this paper, a family of Camassa–Holm equations with distinct parameters is investigated. New solitary wave solutions that include peakons, kinks, compactons, solitary patterns solutions, and plane periodic solutions are formally derived. New schemes that rest mainly on hyperbolic functions are employed to achieve our goal. The work highlights the qualitative change in the physical structures of the obtained solutions.

Keywords: Camassa–Holm family of equations; Peakons; Solitons; Compactons; Kinks

1. Introduction

Camassa and Holm [1] derived a completely integrable wave equation (CH)

$$u_t + 2ku_x - u_{xxt} + auu_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

by retaining two terms that are usually neglected in the small amplitude, shallow water limit. The constant k is related to the critical shallow water wave speed. Eq. (1) can be derived as an asymptotic model for long gravity waves at the surface of shallow water [1–15]. The CH equation, being a model equation for water waves, has its integrable bi-Hamiltonian structure.

For $k = 0$, $a = 3$, it has been shown [1–4] that the CH equation (1) has peaked solitary wave solutions of the form

$$u(x,t) = c\mathrm{e}^{(-|x-ct|)}, \tag{2}$$

where c is the wave speed. The name "peakons", that is, solitary waves with slope discontinuities, was used to single them from general solitary wave solutions since they have a corner at the peak of height c. However, for $k \neq 0$, $a \neq 0$, Qian and Tang [12] investigated the CH equation and obtained two peakons of the form

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$$u(x,t)=\frac{6k}{3-a}\exp\left(-\sqrt{\frac{a}{3}}\left|x-\frac{6kt}{3-a}\right|\right),\quad a\neq 3 \tag{3}$$

and

$$u(x,t)=\frac{2k}{1+a}\left(3\exp\left(-\sqrt{\frac{a}{3}}\left|x-\frac{2kt}{1+a}\right|\right)-2\right). \tag{4}$$

The last peaked solitary wave (4) works for every a, $a>0$.

Nonlinear partial differential equations with dispersion and dissipation effects, that arise in scientific applications, have been under huge size of investigations. Many powerful methods, such as Bäcklund transformation, inverse scattering method, Hirota bilinear forms, pseudo spectral method, the tanh–sech method, the sine–cosine method [15–20], and many others were successfully used to investigate these types of equations. Practically, there is no unified method that can be used to handle all types of nonlinearity.

It is the objective of this work to further complement previous studies to make a further progress in this field. In this work we will study a family of Camassa–Holm equations of the form

$$u_t-u_{xxt}+au_x+buu_x=ku_xu_{xx}+uu_{xxx} \tag{5}$$

and the nonlinearly dispersive integrable equation

$$u_t-k_1u_{xxt}+a_1u_x+\frac{3}{k_1}uu_x=2u_x+b_1u_{xxx}+uu_{xxx}, \tag{6}$$

where a, b, k, a_1, b_1, and k_1 are constants, and $u(x,t)$ is the unknown function depending on temporal variable t and spatial variable x [1–6]. It is obvious that these two equations contain both linear dispersion terms u_{xxt} and u_{xxx}, and the nonlinear dispersion term uu_{xxx}. Eq. (6) appeared first in the works of Fuchssteiner [7], but Camassa and Holm [1] rederived it for certain values of the coefficients. Moreover, this equation can be regarded as an integrable perturbation of the well-known BBM equation [5,6]

$$u_t+u_x+uu_x-u_{xxt}=0. \tag{7}$$

Several forms of the CH equation (5) have been investigated in the literature for several values of the constants. For $b=3$, $k=2$, Eq. (5) reduces to the completely integrable wave Camassa–Holm (CH) equation

$$u_t-u_{xxt}+au_x+3uu_x=2u_xu_{xx}+uu_{xxx}, \tag{8}$$

that describes the unidirectional propagation of shallow water waves over a flat bottom and possesses peakons solutions if $a=0$. As stated before, the CH equation is bi-Hamiltonian and has an infinite number of conservation laws.

For $b=4$, $k=3$, Eq. (5) becomes the second completely integrable Degasperis–Procesi (DP) equation [9]

$$u_t-u_{xxt}+au_x+4uu_x=3u_xu_{xx}+uu_{xxx}, \tag{9}$$

that also possesses peakons solutions if $a=0$.

However, for $a=1$, $b=1$, $k=3$, Eq. (5) is the Fornberg–Whitham (FW) equation

$$u_t-u_{xxt}+u_x+uu_x=3u_xu_{xx}+uu_{xxx}, \tag{10}$$

that appeared in the study of the qualitative behaviors of wave-breaking [2]. This equation admits a peaked solution of the form

$$u(x,t)=Ae^{\left(-\frac{1}{2}\left|x-\frac{4}{3}t\right|\right)}. \tag{11}$$

Moreover, for $b=3$, $k=2$, and a is replaced by $2a$ we obtain the Fuchssteiner–Fokas–Camassa–Holm (FFCH) equation

$$u_t-u_{xxt}+2au_x+3uu_x=2u_xu_{xx}+uu_{xxx} \tag{12}$$