Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 182 (2006) 492–497
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

High-order convergence of the k-fold pseudo-Newton's irrational method locating a simple real zero

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Abstract

By extending the classical Newton's irrational method defined by an iteration

$$x_{n+1} = x_n - f(x_n) \Big/ \sqrt{f'(x_n)^2 - f(x_n)f''(x_n)},$$

we present a high-order k-fold pseudo-Newton's irrational method for locating a simple zero of a nonlinear equation. Its order of convergence is proven to be at least $k + 3$ and the convergence behavior of the asymptotic error constant is investigated near the corresponding simple zero. A root-finding algorithm is described as well as the introduction on the convergence of the fixed-point iterative method. Various numerical examples have successfully demonstrated a good agreement with the theory presented here.

Keywords: k-Fold pseudo-Newton's irrational method; Ostrowski's method; Order of convergence; Asymptotic error constant

1. Introduction

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and natural numbers, respectively. Assuming that a function $f : \mathbb{R} \to \mathbb{R}$ has a simple real zero α and is sufficiently smooth in a neighborhood of α, we wish to locate α accurately with a high-order method. To this end we first rewrite the equation $f(x) = 0$ in the form $x - g(x) = 0$, where $g : \mathbb{R} \to \mathbb{R}$ is assumed to be sufficiently smooth in a neighborhood of α. Then we seek an approximated α using an iterative method

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.04.010

where $x_0 \in \mathbb{R}$ is given. For a given $p \in \mathbb{N}$, we further assume that

$$\begin{cases} \left|\frac{d^p}{dx^p} g(x)\right|_{x=\alpha} = |g^{(p)}(\alpha)| < 1 & \text{if } p = 1, \\ g^{(i)}(\alpha) = 0 \quad \text{for } 1 \leqslant i \leqslant p-1 \text{ and } g^{(p)}(\alpha) \neq 0 & \text{if } p \geqslant 2. \end{cases} \tag{1.2}$$

Let x_n belong to a sufficiently small neighborhood of α for $n \in \mathbb{N} \cup \{0\}$. Then Taylor series [1,6] expansion about α immediately gives

$$x_{n+1} = g(x_n) = g(\alpha) + g^{(p)}(\xi)(x_n - \alpha)^p/p! \tag{1.3}$$

where $\xi \in (a, b)$ with $a = \min(\alpha, x_n)$ and $b = \max(\alpha, x_n)$. Since g is continuous at α, we find that for all given $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| = |g^{(p)}(\xi)| \frac{|(x_n - \alpha)^{p-1}|}{p!} |x_n - \alpha| < \epsilon, \tag{1.4}$$

whenever $|x_n - \alpha| < \delta$. Let $\boldsymbol{J} = \{x : |x - \alpha| \leqslant \delta\}$. Then the continuity of $g^{(p)}$ on $\boldsymbol{J}$ ensures the existence of a number $M > 0$ satisfying $|g^{(p)}(x)| \leqslant M$ for all $x \in \boldsymbol{J}$. Choose

$$\delta = \begin{cases} \min(\epsilon, 1/M) & \text{if } p = 1, \\ \{\min(\epsilon^{p-1}, p!/M)\}^{1/(p-1)} & \text{if } p \geqslant 2. \end{cases}$$

Then $|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| \leqslant |x_n - \alpha|$. Hence, $g : \boldsymbol{J} \to \boldsymbol{J}$. Since $|x_n - \alpha| < \delta$, it follows from (1.4) that

$$|x_{n+1} - \alpha| \leqslant |g(x_n) - g(\alpha)| \leqslant K|x_n - \alpha|, \tag{1.5}$$

where $0 < K = \sup\{M|(x_n - \alpha)|^{p-1}/p! : n \in \mathbb{N} \cup \{0\}\} < M\delta^{p-1}/p! \leqslant 1$ for $p \geqslant 2$. If $p = 1$, then $K = M < 1$ can be chosen according to (1.2). Hence g is contractive on $\boldsymbol{J}$ for any $p \in \mathbb{N}$ and the sequence $\{x_n\}_{n=0}^{\infty}$ with $x_0 \in \boldsymbol{J}$ defined by (1.1) converges to a fixed point $\alpha \in \boldsymbol{J}$ [7]. Now introducing $e_n = x_n - \alpha$ with the fact that $\lim_{n\to\infty} \xi = \alpha$, for the iterative method (1.1) we obtain the *asymptotic error constant* η and *order of convergence* p [2,7] as follows:

$$\eta = \lim_{n\to\infty} \left|\frac{e_{n+1}}{e_n^p}\right| = |g^{(p)}(\alpha)|/p!. \tag{1.6}$$

Now for an arbitrarily given $x \in \mathbb{R}$, we define a function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(w) = w \pm f(w) \Big/ \sqrt{f'(x)^2 - f(w)f''(x)} \tag{1.7}$$

with the sign $\pm$ as that of $-f'(\alpha)$. Here, we take minus sign for ease of analysis. We denote

$$w_0 = F(x) = x - f(x) \Big/ \sqrt{f'(x)^2 - f(x)f''(x)}$$

and define a function

$$w_k(x) = F(w_{k-1}) = w_{k-1} - f(w_{k-1}) \Big/ \sqrt{f'(x)^2 - f(w_{k-1})f''(x)} \tag{1.8}$$

iteratively for $k \in \mathbb{N}$. Hence, $w_k(x) = F^k(w_0) = F^{k+1}(x)$ for $k \in \mathbb{N}$, where $F^k(w_0) = F(F(\cdots F(w_0)\cdots))$. Then the iterative method with $x_0 \in \mathbb{R}$

$$x_{n+1} = F^{k+1}(x_n) = g(x_n) \tag{1.9}$$

is called the *k-fold pseudo-Newton's irrational method(pseudo-Ostrowski's method)*. If $k = 0$, it is called the *Newton's irrational method* or *Ostrowski's method* and has the cubic convergence as shown in Laguerre-type numerical methods [4,5] including Halley's method and leap-frogging Newton's method [3]. If $k = 1$, it is simply called the *pseudo-Newton's irrational method*.