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On the expected optimal value and the optimal expected value

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Abstract

Approaches to stochastic optimization have followed a variety of modeling philosophies, but little has been done to systematically compare different models found in the literature. This article is concerned with the basic concepts (and a comparison between them) underlying optimality under uncertainty, which is ubiquitous in all realistic problems of science and engineering. Specifically, it discusses two basic ideas—the minimum (maximum) expected value criterion and the expected minimum (maximum) value criterion—in a theoretical context. Illustrative applications are presented to justify the theoretical results.

Keywords: Uncertainty; Random variable; Probability; Expected value; Optimal point

1. Introduction

Problems of optimality under uncertainty occur frequently in a wide variety of real-world problems in science, engineering and technology, which have probabilistic parameters, nondeterministic initial conditions, uncertain input situations and models based on incomplete knowledge. Specifically, a large number of problems such as engineering design [1], supply-allocation [2], production planning [3] and scheduling [4], transportation [5], inventory network [6,7], finance [8], require that decisions be made in the presence of uncertainty. Uncertainty, for instance, governs the prices of fuels, the availability of electricity, and the demand for chemicals. In other words, much of life involves making optimal choices under uncertainty, i.e., choosing the optimal from some set of optional courses of action in uncertain situations. Several approaches to stochastic optimization have been proposed [1,9–12]. Sahinidis [13] discusses the state of the art in optimization under uncertainty, and notes the need for a systematic comparison of current methodologies.

The work of Huyse [1], which actually motivated our investigation, deals with the problem of airfoil shape optimization. It requires the calculation of an optimum shape under the parametric uncertainty associated with Mach number. They employ the MEV concept that optimizes the expectation of the objective/risk function subject to some expectation constraint. There is no reason why one should not consider the expectation of

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the optimized objective/risk function (EMV). Thus our study focuses, in a straightforward fashion, on a comparison between the EMV and MEV approaches.

The theory developed in the next two sections shows that the expectation and optimization operators are not commutative and that EMV $\leqslant$ MEV for continuous objective functions. Moreover, if the randomness is nonsymmetric or the functional to be minimized is nonlinear, the probability of having a lower risk (lower value of the objective function) at the EMV design point compared to the MEV design strategy is greater than 50%. Applications to simple stochastic problems of free undamped vibrations, population growth, and nonlinear Burgers equation are considered, which verify the theoretical results.

2. Statement of the problem

Let $(\Omega, \mathscr{F}, P)$ be a probability space and let ω be an $\mathscr{F}$-measurable random variable. We denote by d the design variable, whose domain is confined to a Banach space $(X, \|\cdot\|)$. Let us consider a functional $J : X \times \Omega \to \mathbb{R}$, which is assumed to have a minimum for any given value of the random variable ω. The minimization problem under uncertainty consists in finding an optimal point $d^* \in D \subseteq X$, such that

$$J(d^*, \omega) = \min_{d \in D} J(d, \omega), \quad \text{for all } \omega \in \Omega. \tag{1}$$

The problem of optimization under uncertainty (1) is quite challenging because it requires finding an optimum point that minimizes the functional $J(\cdot, \omega)$ for all possible values of the random variable ω. In general, one may find different optimum design variables associated with different values of the random variable. In other words, for different values ω_1, ω_2 of the random variable ω, the corresponding optima $d^*(\omega_1)$, and $d^*(\omega_2)$ may be different.

Once randomness is included in the formulation of the mathematical problem, it is not immediately clear how to formulate a well-posed optimization problem. A number of convenient formulations of the problem (1) are possible depending on when decisions must be taken relative to the realization of the random variable. In the approach used and justified in [1], the best design or decision is the one which minimizes the overall risk, and it is based on the Von Neumann–Morgenstern statistical decision theory [14]. In this case, the optimization problem is posed as

$$\min_{d \in D} E(J(d, \omega))$$

and the Bayes' decision is given by

$$d_{\mathrm{MEV}} \stackrel{\text{def.}}{=} \arg\min_{d \in D} E(J(d, \omega)). \tag{2}$$

On the other hand, a practical solution to the minimization problem (1) may well be to "average" over the entire range of optima, i.e., to consider the optimization problem and the optimum solution respectively to be

$$E\left(\min_{d \in D} J(d, \omega)\right)$$

$$d_{\mathrm{EMV}} \stackrel{\text{def.}}{=} E\left(\arg\min_{d \in D} J(d, \omega)\right). \tag{3}$$

In our opinion, this new formulation (3), which we call the expected minimum value criterion (EMV), is a more natural representation of the original problem under consideration (1). It is therefore of interest to examine how the EMV solution d_{EMV} compares with the MEV solution d_{MEV}. It is quite clear that the EMV criterion is rather closer to the initial optimization problem, but does it give us a "better" optimum? The purpose of the present article is to address this question, and to prove and exemplify the advantages of one versus the other. The next section derives some theoretical results, and the following one presents some simple examples in support of the theoretical results.

3. Theoretical results

This section concerns the comparison of the two optimum design points proposed earlier. Let us remark first that the following inequality holds for any continuous objective function.