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Quadratic non-polynomial spline approach to the solution of a system of second-order boundary-value problems

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Abstract

A quadratic non-polynomial spline functions based method is developed to find approximations solution to a system of second-order boundary-value problems associated with obstacle, unilateral, and contact problems. The present approach has less computational cost and gives better approximations than those produced by other collocation, finite-difference and spline methods. Convergence analysis of the method is discussed. A numerical example is given to illustrate practical usefulness of the new method.

Keywords: Non-polynomial splines; Quadratic spline functions; Finite-difference methods; Obstacle problems; Boundary-value problems

1. Introduction

Variational inequality theory has become an effective and powerful tool for studying obstacle and unilateral problems arising in mathematical and engineering sciences. This theory has developed into an interesting branch of applicable mathematics, which contains a wealth of new ideas for inspiration and motivation to do research. It has been shown by Kikuchi and Oden [8] that the problem of equilibrium of elastic bodies in contact with a rigid foundation can be studied in the framework of variation inequality theory. In variational inequality formulation, the location of free boundary (contact problem) becomes an intrinsic part of the solution and no special techniques are needed to locate it. Various numerical methods are being developed and applied to find the numerical solutions of the obstacle problems including finite difference techniques and spline based methods. In principle, these methods cannot be applied directly to solve the obstacle problems. However, if the obstacle function is known, one can characterize the obstacle problem by a sequence of

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boundary-value problems without constraints via the variational inequality and penalty function. The computational advantage of this approach is its simple applicability for solving differential equations. Such types of penalty function methods have been used quite effectively by Noor and Tirmzi [10], as a basis for obtaining numerical solutions for some obstacle problems.

In this paper, non-polynomial spline functions are used to develop a numerical method for obtaining smooth approximations to the solution of a system of second-order boundary-value problems of the type,

$$y'' = \begin{cases} f(x), & a \leqslant x < c, \\ g(x)y(x) + f(x) + r, & c \leqslant x < d, \\ f(x), & d \leqslant x \leqslant b \end{cases} \tag{1.1}$$

with the boundary conditions,

$$y(a) = \alpha_1 \quad \text{and} \quad y(b) = \alpha_2 \tag{1.2}$$

and the continuity conditions of y and y' at c and d. Here, f and g are continuous functions on $[a,b]$ and $[c,d]$ respectively. The parameters α_1, α_2, and r, are real finite constants. Such type of systems arises in the study of obstacle, unilateral, moving and free boundary value problems, see, for example [1–6,8] and the references therein. In general it is not possible to obtain the analytical solution of (1.1) for arbitrary choices of $f(x)$ and $g(x)$, we usually resort to some numerical methods for obtaining an approximate solution of (1.1).

Noor and Khalifa [9] have solved problem (1.1) using collocation method with cubic splines as basis functions. They have shown that this collocation method gives approximation with first order-accuracy. Similar conclusions were pointed out by Noor and Tirmzi [10], where second-order finite difference methods were used to solve problem (1.1). On the other hand, Al-Said [1,2] has developed and analyzed quadratic and cubic splines for solving (1.1). He proved that both quadratic and cubic splines methods can be used to produce second-order smooth approximation for the solution of Eq. (1.1) and its first derivative over the whole range of integration. More recently, Siraj-ul-Islam et al. [11–13] have established and analyzed optimal smooth approximations for systems second, third-order boundary value problems and a class of methods for special fourth-boundary-value problems based on cubic, quartic and sextic non-polynomial splines and which provides bases for this approach.

In the present paper, quadratic non-polynomial spline functions are applied to develop a new numerical method for obtaining smooth approximations to the solution of such system of second-order differential equations. The new method is of order two for arbitrary α and β if $2\alpha + 2\beta - 1 = 0$ and method is of order four if $\alpha = \frac{1}{12}$ along with $2\alpha + 2\beta - 1 = 0$. As evident form [1,2] the quadratic and cubic polynomial spline functions need three and four coefficients evaluation at each subinterval and can produce a numerical scheme of second-order accuracy where as quadratic non-polynomial spline functions need three coefficients evaluation at each subinterval $[x_i, x_{i+1}]$ and produces a fourth-order scheme with a less computational cost. This improvement is because of introduction of parameter k in the trigonometric part of T_2 given below. The spline function we propose in this paper has the form $T_2 = \mathrm{Span}\{1, \cos kx, \sin kx\}$ where k is the frequency of the trigonometric part of the splines function which can be real or pure imaginary and which will be used to raise the accuracy of the method. Thus in each subinterval $x_i \leqslant x \leqslant x_{i+1}$, we have

$$\begin{aligned} &\mathrm{Span}\{1, \sin x, \cos\ x\}, \\ &\mathrm{Span}\{1, \sinh x, \cosh x\} \quad \text{or} \\ &\mathrm{Span}\{1, x, x^2\}, \quad (\text{when } k = 0). \end{aligned}$$

This fact is evident when correlation between polynomial and non-polynomial spline bases functions is investigated in the following manner:

$$T_2 = \mathrm{Span}\{1, \cos kx, \sin kx\} = \mathrm{Span}\left\{1, \frac{1}{k}\sin(kx), \frac{2}{k^2}(1 - \cos(kx))\right\}. \tag{1.3}$$

Form Eq. (1.3), it follows that $\mathrm{Lt}_{k\to 0} T_2 = \{1, x, x^2\}$.

The main idea is to use the condition of continuity to get recurrence relation for (1.1). The advantage of new method is higher accuracy with the less computational effort. In comparison with the finite difference