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Bifurcations of limit cycles in a cubic system with cubic perturbations ☆

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Abstract

This paper is concerned with limit cycles on two different cubic systems with nine singular points. Eleven limit cycles are found and the distributions are studied by using the methods of bifurcation theory and qualitative analysis.

Keywords: Limit cycle; Homoclinic loop; Heteroclinic loop; Hamiltonian system

1. Introduction

There have been many studies on the number and distributions of limit cycles for planar polynomial systems (see [1–18] for example). There are three main aspects to study the number of limit cycles for perturbations of Hamiltonian systems as follows. These are:

(1) *Hopf bifurcation*. The main idea is to compute Liapunov constants or focus values for a focus and then change the stability of the focus by varying focus values to obtain limit cycles one by one (see [1,9,14,15]). The maximal number of limit cycles which can appear in a neighborhood of the focus under perturbations is called the Hopf cyclicity of the focus about the considered system. The Hopf cyclicity can also be obtained by coefficients appearing in the expression of the first Melnikov function near a center (see [3]).

(2) *Poincare bifurcation*. The main task in Poincare bifurcation is to find the maximal number of limit cycles generated by members of a family of periodic orbits. There have been a lot of studies on this aspect (see [9–12,14,15]). The number can be obtained through the study of the total number of zeros for the first or higher order Melnikov functions using the method of geometric analysis. Using this method eleven limit cycles were obtained in [10–12] for the following cubic systems:

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$$\dot{x} = y(1 - cy^2) + \mu x(mx^2 + ny^2 - \lambda),$$
$$\dot{y} = -x(1 - ax^2) + \mu y(mx^2 + ny^2 - \lambda),$$

with $a > c > 0$, $0 < \mu \ll 1$, and

$$\dot{x} = y(1 + x^2 - ay^2) + \varepsilon x(mx^2 + ny^2 - \lambda),$$
$$\dot{y} = -x(1 - cx^2 + y^2) + \varepsilon y(mx^2 + ny^2 - \lambda),$$

with $a > c > 0$, $ac > 1$, $0 < \varepsilon \ll 1$.

(3) *Homoclinic or heteroclinic bifurcation*. In 1986, Roussarie [13] originated a way to study the number of limit cycles appearing near a homoclinic loop and obtained a significant result to find the homoclinic cyclicity by using coefficients in the expression of the first order Melnikov function. Then the idea was developed and more results were obtained in [2,4–6]. It consists of the following three steps:

- find discriminate values to determine the stability of the homoclinic or a double homoclinic loop;
- vary parameters to change the stability of the loop to produce limit cycles;
- find a final limit cycle by breaking the homoclinic loop.

Recently, eleven limit cycles were found near double homoclinic loops by using this method in [8,16] and a new distribution of 11 limit cycles was found. The systems studies in [8,16] are, respectively,

$$\dot{x} = y(1 - y^2) + \mu P_3(x, y),$$
$$\dot{y} = -x(1 - x^2) + \mu Q_3(x, y),$$
$$\dot{x} = y(1 - cy^2) + \mu(a_{10}x - a_{10}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{01}y + a_{03}y^3),$$
$$\dot{y} = -x(1 - x^2) + \mu(b_{01}y + b_{21}x^2y + b_{12}xy^2 + b_{10}x - b_{10}x^3 + b_{03}y^3),$$

with $c > 1$, $0 < \mu \ll 1$, and

$$\dot{x} = y(1 + x^2 + cy^2) + \varepsilon\left(a_{10}x + a_{01}y + \sum_{i+j=3} a_{ij}x^iy^j\right),$$
$$\dot{y} = x(1 - ax^2 - y^2) + \varepsilon\left(b_{10}x + b_{01}y + \sum_{i+j=3} b_{ij}x^iy^j\right),$$

with $0 > c > -1 > a$, $ac > 1$.

We consider the following perturbed cubic Hamiltonian system

$$\begin{aligned} \dot{x} &= y(1 + x^2 - ay^2) + \varepsilon\left(a_{10}x + a_{01}y + \sum_{i+j=3} a_{ij}x^iy^j\right), \\ \dot{y} &= -x(1 - cx^2 + y^2) + \varepsilon\left(b_{10}x + b_{01}y + \sum_{i+j=3} b_{ij}x^iy^j\right), \end{aligned} \tag{1.1}$$

where a and c are positive constants and $\varepsilon > 0$ is small. Without loss of generality, we may assume $a \geqslant c > 0$, $ac > 1$. Also, we consider the coefficients a_{ij} and b_{ij} in (1.1) as parameters. Our main result can be stated as the conclusion in Section 5 and the distributions of limit cycles see Fig. 1.1.

In Section 2, we give the phase portraits of the unperturbed system. In Section 3, we verify the results following the technique developed in [2,5,8,16] and consider the case of $a \neq c$ and $a = c$ separately. For convenience of numerical analysis we will suppose $a = 2$ and $c = 1$ for the first case and $a = c = 2$ for the second.