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Analysis of three-dimensional grids: The cube and the octahedron

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Abstract

The analysis of three-dimensional data is customarily performed by arranging eight measurements in a cubical array and representing them by the trilinear equation. The octahedral design offers the prospect of reduced laboratory costs because it requires only six measurements. Operational interpolating equations for six data in octahedral array can be more accurate than the trilinear equation for eight data in cubical array. The operational equations apply to the six- or seven-point octahedron.

Keywords: Interpolation; Response surfaces; Cubic design; Octahedral design; Operational equations

1. Introduction

The eight-point cube is a design that is used for the analysis of a phenomenon that depends on three independent parameters. Eight measurements are obtained, each one representing a different combination of the high and low levels of the parameters. The data are arranged at the vertices of a cube. They are then represented by the trilinear equation. That equation does not estimate quadratic coefficients but it estimates a coefficient that represents the simultaneous effect of all three parameters.

In recent years, new equations for treating eight or nine data in cubical array have appeared. They assume polynomial, exponential, or trigonometric forms [1]. The equations estimate linear, cross-product, and curvature coefficients [2]. Many of the equations can accommodate a measurement at the center point of the cube, something that is not possible when applying the trilinear equation [3–5].

In their eight-point forms, the new equations use the same data that are used by the trilinear equation. Data that are suitable for the trilinear equation can be represented by operational equations at no additional cost. This convenience suggests an economic advantage because more experimental work may not be needed for curvature-term estimations. However, laboratory expenses remain the same as long as the eight-point cube is the standard design for experiments that depend on three independent parameters.

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2. Three equations for the octahedron

The purpose of this paper is to suggest the octahedron as a potential alternative to the cube. The cube requires a minimum of eight experiments whereas the octahedron requires only six experiments. The 25% reduction in laboratory costs promised by octahedral designs may be appealing. This reduction is accompanied by a reduction in information about the system but there may be situations in which sacrificing information for reduced expenses can be justified.

A skeleton illustration of the octahedral design appears in Fig. 1. It is formed by the intersection of three mutually perpendicular, equal lines. A measurement can be made at the end of each line in Fig. 1. The data are denoted A, B, C, E, F, G as in the figure. In the $-1..1$ coordinate system, the data have coordinates $(0,0,-1)$, $(0,-1,0)$, $(-1,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, respectively. These six measurements can be supplemented by an optional datum, D, at the center point of the design. It that case, the design represents a seven-point octahedron. Measurement D is located at $(0,0,0)$.

The analysis of four- and five-point diamond arrays has been illustrated in recent times [6,7]. A six-point octahedron is formed by three mutually perpendicular, four-point diamond arrays. They occur in the x,y-, x,z-, and y,z-planes. Three mutually perpendicular, five-point diamond arrays are formed if a measurement is obtained at the center point D of the octahedral array.

The first equation that interpolates the three, four-point diamonds that generate an octahedron is based on Eq. (5) in Ref. [6]. The cross-product coefficients in the first equation for the octahedron do not utilize an estimated or measured datum D at the center point of the design. These three coefficients terms are listed as Eqs. (1)–(3) below (see Fig. 1).

$$xyc = (C-E)(B-F)(F+B-E-C)/[(B+C-F-E)(B-C-F+E)], \quad (1)$$

$$xzc = (C-E)(A-G)(G+A-E-C)/[(A+C-G-E)(A-C-G+E)], \quad (2)$$

$$yzc = (B-F)(A-G)(G+A-F-B)/[(A+B-G-F)(A-B-G+F)]. \quad (3)$$

An expression for the center point of a four-point diamond array appears as Eq. (2) in Ref. [6]. In the case of three mutually perpendicular, four-point diamond arrays, the center point is response estimated by Eq. (4) below:

$$\begin{aligned} D = {} & [(F-B)^2(E+C)-(E-C)^2(F+B)]/[6(B-C+E-F)(B+C-E-F)] \\ & + [(G-A)^2(E+C)-(E-C)^2(G+A)]/[6(A-C+E-G)(A+C-E-G)] \\ & + [(G-A)^2(F+B)-(F-B)^2(G+A)]/[6(A-B+F-G)(A+B-F-G)]. \end{aligned} \quad (4)$$

The first interpolating equation for the octahedron illustrated in Fig. 1 is based on Eqs. (1)–(4). It appears as Eq. (5). The letter R represents the interpolated value for any combination of the x-, y-, and z-coordinates in the $-1..1$ coordinate system. The subscript 1, as in R_1, means that Eq. (5) is the first operational equation for the six-point octahedral design. D is estimated by means of Eq. (4).

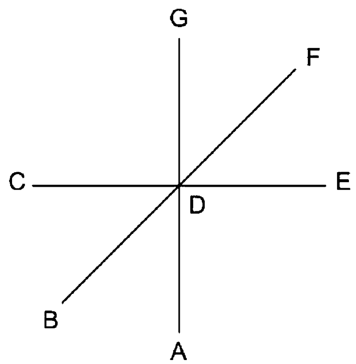

Fig. 1. Skeleton of the octahedral design. Only the three mutually-perpendicular axes are shown.