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Uniform difference method for singularly perturbed Volterra integro-differential equations

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Abstract

Singularly perturbed Volterra integro-differential equations is considered. An exponentially fitted difference scheme is constructed in a uniform mesh which gives first order uniform convergence in the discrete maximum norm. Numerical experiments support the theoretical results.

Keywords: Singular perturbation; Volterra integro-differential equations; Difference scheme; Uniform convergence; Boundary layer

1. Introduction

This paper is concerned with the following singularly perturbed Volterra integro-differential equations:

$$\varepsilon u'(t) + a(t)u(t) + \int_0^t K(t,s)u(s)\mathrm{d}s = f(t), \quad t \in I := [0,T], \tag{1.1}$$

$$u(0) = A, \tag{1.2}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, $a(t) \geqslant \alpha > 0$, $f(t)(t \in I)$ and $K(t,s)((t,s) \in I \times I)$ are sufficiently smooth functions and A is a given constant. On putting $\varepsilon = 0$ in Eq. (1.1), we obtain the reduced equation

$$a(t)u_r(t) + \int_0^t K(t,s)u_r(s)\mathrm{d}s = f(t), \tag{1.3}$$

which is a Volterra integral equation of the second kind. The singularly perturbed nature of (1.1) occurs when the properties of the solution with $\varepsilon > 0$ are incompatible with those when $\varepsilon = 0$. The interest here is in those problems which do imply such an incompatibility in the behavior of u near $t = 0$. This suggests the existence of a boundary layer near the origin where the solution undergoes a rapid transition.

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Singularly perturbed Volterra integro-differential equations arise in many physical and biological problems. Among these are diffusion–dissipation processes, epidemic dynamics, synchronous control systems, renewal processes and filament stretching (see, e.g., [6,7,11,12,16]).

In Ref. [16], many qualitative properties of the solution of a nonlinear singularly perturbed Volterra integro-differential equation is established and existence and uniqueness of the solution of equation are shown under some assumptions. In Refs. [11,12], the result of [16] is extended to the general nonconvolution problem. In Refs. [6,7], a formal methodology for obtaining asymptotic expansions of solutions of singularly perturbed Volterra integro-differential equations is developed and applied to several example problems. In Ref. [5], a formal asymptotic scheme is used to determine the leading order behavior of a certain singularly perturbed integro-differential equation which models the process of stretching a polymer filament.

Implicit Runge–Kutta methods were analyzed for singularly perturbed integro-differential–algebraic equations in Ref. [13] and for singularly perturbed integro-differential systems in Ref. [14]. It was supposed that the initial value of the reduced equation lies in the smooth outer solution. Therefore, in Refs. [13,14] the convergence of the implicit Runge–Kutta methods has been proved out of the boundary layer. A survey of the existing literature on singularly perturbed Volterra integral and integro-differential equations are given in Ref. [15]. The exponential scheme that has a fourth order accuracy when the perturbation parameter ε, is fixed, is derived and stability analysis of this scheme is discussed in Ref. [19]. The numerical discretization of singularly perturbed Volterra integro-differential equations and Volterra integral equations by tension spline collocation methods in certain tension spline spaces are considered in Ref. [10].

A vast review of the literature on numerical methods for singularly perturbed differential equations may be found in Refs. [8,9,17,18].

Here we analyze an exponentially fitted difference scheme on a uniform mesh for the numerical solution of problem (1.1) and (1.2). The difference scheme is constructed by the method integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form [1–4].

The paper is organized as follows: In Section 2, the asymptotic estimations of the problem (1.1) and (1.2) are established. In Section 3, the difference scheme constructed on the uniform mesh for numerical solution (1.1) and (1.2) is presented. Stability and convergence of the difference scheme are investigated in Section 4 and error of the difference scheme is evaluated in Section 5. A numerical example is also taken up in Section 6.

Notation. Let

$$\omega_h = \left\{ t_i = ih, i = 1, 2, \ldots, N; h = \frac{T}{N} \right\}, \quad \varpi_h = \omega_h \cup \{t = 0\}$$

be the uniform mesh on $[0, T]$.

Here and throughout the paper we use the notation

$$g_{\bar{t},i} = \frac{g_i - g_{i-1}}{h}, \quad g_{i-1/2} = g(t_i - h/2),$$

$g_i = g(t_i)$ for any continuous function $g(t)$.

In our estimates we use the maximum norm given by

$$\|g\|_\infty = \max_{[0,T]} |g(t)|.$$

For any discrete function g_i we also define the corresponding discrete norm by

$$\|g\|_{\infty,\omega_h} \equiv \|g\|_\infty = \max_{1\leqslant i\leqslant N} |g_i|.$$

Throughout the paper, C, sometimes subscripted, a generic positive constant that is independent of ε and any mesh is used. A subscripted C is fixed in value throughout the paper. $C^n(I \times I)$ denotes the space of real-valued functions which are n-times continuously differentiable on $I \times I$. $C^n_m(I \times I)$ denotes the space of two-valued functions which are n-times continuously differentiable with respect to the first argument and m-times continuously differentiable with respect to the second argument on $I \times I$.