Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Applied Mathematics and Computation 177 (2006) 589–596
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Bifurcations of travelling wave solutions in a class of generalized KdV equation ☆

Shengqiang Tang [a,*], Ming Li [b]

[a] Department of Computing Science and Mathematics, Guilin University of Electronic Technology, Guilin, Guangxi 541004, PR China
[b] Department of Mathematics, Qujin Normal Institute Qujin, Yunnan 655000, PR China

Abstract

By using the bifurcation theory of dynamical systems to a generalized KdV equation given by Cooper et al., the existence of solitary wave solutions, solitary cusp wave solutions and uncountably infinite many smooth and non-smooth periodic wave solutions is obtained. Under different parametric conditions, various sufficient conditions to guarantee the existence of the above solutions are given. Some exact explicit parametric representations of the above waves are determined.

Keywords: Solitary wave solution; Periodic wave solution; Smoothness of wave; Generalized KdV equation

1. Introduction

In 1993, Cooper et al. [2] considered the following generalized KdV equation:

$$K^*(l,p) : u_t = u_x u^{l-2} + \alpha[2u_{xxx}u^p + 4pu^{p-1}u_x u_{xx} + p(p-1)u^{p-2}(u_x)^3], \tag{1.1}$$

where $p, l \geqslant 2$, $l, p \in Z^+$. This equation was derived from the Lagrangian

$$L(l \cdot p) = \int \left[\frac{1}{2}\psi_x\psi_t - \frac{(\psi_x)^l}{l(l-1)} + \alpha(\psi_x)^p(\psi_{xx})^2\right] \mathrm{d}x, \tag{1.2a}$$

where $u(x,t)$ was defined by $u(x,t) = \psi_x(x,t)$. The authors of [2] investigated the Hamiltonian structure and integrability properties for this class of KdV equation. It is very important to consider the dynamical bifurcation behavior for the travelling wave solutions of (1.1). In this paper, we shall study all travelling wave solutions in the parameter space of this system. Let $u(x,t) = \phi(x - ct) = \phi(\xi)$, where c is the wave speed. Then (1.1) becomes

☆ This research was supported by the Science Foundation of Guangxi province (0575092).
* Corresponding author.
E-mail address: tangsq@gliet.edu.cn (S. Tang).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2005.09.089

$$-c\phi' = \frac{1}{l-1}(\phi^{l-1})' + \alpha[2(\phi^p\phi'')' + p(\phi^{p-1}(\phi')^2)'], \tag{1.2b}$$

where "′" is the derivative with respect to ξ. Integrating (1.2b) once and setting two integration constants as g, we have

$$2\alpha\phi^p\phi'' + \alpha p\phi^{p-1}(\phi')^2 + \frac{1}{l-1}\phi^{l-1} + c\phi + g = 0. \tag{1.3}$$

Eq. (1.3) is equivalent to the two-dimensional systems as follows for $g = 0$:

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = -\frac{\alpha p\phi^{p-2}y^2 + \frac{1}{l-1}\phi^{l-2} + c}{2\alpha\phi^{p-1}} \tag{1.4a}$$

and for $g \neq 0$,

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = -\frac{\alpha p\phi^{p-1}y^2 + \frac{1}{l-1}\phi^{l-1} + c\phi + g}{2\alpha\phi^{p}} \tag{1.4b}$$

with the first integral

$$H(\phi, y) = \alpha\phi^p y^2 + \frac{1}{l(l-1)}\phi^l + \frac{1}{2}c\phi^2 + g\phi = h. \tag{1.5}$$

System (1.4b) is a 5-parameter planar dynamical system depending on the parameter group (l, p, α, c, g). For different l, p and a fixed α, we shall investigate the bifurcations of phase portraits of (1.4b) in the phase plane (ϕ, y) as the parameters c, g are changed. Here we are considering a physical model where only bounded travelling waves are meaningful. So we only pay attention to the bounded solutions of (1.4b).

Suppose that $\phi(\xi)$ is a continuous solution of (1.4b) for $\xi \in (-\infty, \infty)$ and $\lim_{\xi\to\infty}\phi(\xi) = a$, $\lim_{\xi\to-\infty}\phi(\xi) = b$. Recall that (i) $\phi(x,t)$ is called a solitary wave solution if $a = b$; (ii) $\phi(x,t)$ is called a kink or anti-kink solution if $a \neq b$. Usually, a solitary wave solution of (1.1) corresponds to a homoclinic orbit of (1.4b); a kink (or anti-kink) wave solution (1.1) corresponds to a heteroclinic orbit (or the so-called connecting orbit) of (1.4b). Similarly, a periodic orbit of (1.4b) corresponds to a periodically travelling wave solution of (1.1). Thus, to investigate all possible bifurcations of solitary waves and periodic waves of (1.1), we need to find all periodic annuli and homoclinic orbits of (1.4b), which depend on the system parameters. The bifurcation theory of dynamical systems (see [1,3]) plays an important role in our study.

We notice that the right hand of the second equation in (1.4b) is not continuous when $\phi = 0$. In other words, on the above straight line of the phase plane (ϕ, y), ϕ''_ξ has no definition. It implies that the smooth system (1.1) sometimes has non-smooth travelling wave solutions. This phenomenon has been studied by some authors (see [3,4]). we claim that the existence of a singular straight line for a travelling wave equation is the original reason why travelling waves lose their smoothness.

The paper is organized as follows. In Section 2, we discuss bifurcations of phase portraits of (1.4b). In Section 3, some explicit parametric representations of travelling wave solutions are given. In Section 4, the existence of smooth solitary wave solutions and uncountable infinite many non-smooth periodic wave solutions of (1.1) is discussed.

2. Bifurcations of phase portraits of (1.4b)

In this section, we study all possible periodic annuluses defined by the vector fields of (1.4b) when the parameters c, g are varied. Throughout we assume that $\alpha > 0$. Otherwise, we can make a transformation to reduce (1.4b) to this case.

Let $d\xi = \alpha\phi^p d\zeta$. Then, except on the straight lines $\phi = 0$, the system (1.4b) has the same topological phase portraits as the following system:

$$\frac{d\phi}{d\zeta} = 2\alpha\phi^p y, \qquad \frac{dy}{d\zeta} = -\left[\alpha p\phi^{p-1}y^2 + \frac{1}{l-1}\phi^{l-1} + c\phi + g\right]. \tag{2.1}$$