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Surfaces with common geodesic in Minkowski 3-space

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Abstract

In this paper, we analyzed the problem of constructing a family of surfaces from a given spacelike (or timelike) geodesic curve. Using the Frenet trihedron frame of the curve in Minkowski space, we express the family of surfaces as a linear combination of the components of this frame, and derive the necessary and sufficient conditions for the coefficients to satisfy both the geodesic and the isoparametric requirements. Finally, examples are given to show the family of surfaces with common geodesic.

Keywords: Minkowski space; Geodesic curve; Ruled surface

1. Introduction

Geodesics are important in the relativistic description of gravity. Einstein's principle of equivalence tells us that geodesics represent the paths of freely falling particles in a given space. (Freely falling in this context means moving only under the influence of gravity, with no other forces involved.)

The geodesics principle states that the free trajectories are the geodesics of space. It plays a *very important role in a geometric-relativity theory*, since it means that the fundamental equation of dynamics is completely determined by the geometry of space, and therefore has not to be set as an independent equation. Moreover, in such a theory the action identifies (up to a constant) with the fundamental length invariant, so that the stationary action principle and the geodesics principle become identical.

On a Minkowski surface, tangent vectors are classified into timelike, null, or spacelike, and so a (smooth) curve on the surface is said to be timelike, null, or spacelike if its tangent vectors are always timelike, null, or spacelike, respectively. In fact, a timelike curve corresponds to the path of an observer moving at less than the speed of light. Null curves correspond to moving at the speed of light, and spacelike curves to moving faster than light.

Geodesics are curves along which geodesic curvature vanishes. This is of course where the geodesic curvature has its name from. Since Lorentzian metric is not positive definite metric, the distance function dS^2 can be

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positive, negative or zero, whereas the distance function in the Euclidean space can only be positive. Thus, we have to separate our geodesics on the basis of whether the distance function is positive, negative or zero. Geodesics with $dS^2 < 0$ are called spacelike geodesics. Geodesics with $dS^2 > 0$ are called timelike geodesics. Geodesics with $dS^2 = 0$ are called null geodesics.

On a Riemannian space, with a positive definite metric (so that all directions are spacelike rather than timelike), the shortest curve between two points is a geodesic, so it cannot be shortened. But in Minkowski space, any spacelike curve can be shortened by bending it in the timelike direction so that it is more nearly lightlike or null (in the limit of which it would have zero length).

Surfaces with common geodesic have been the subject of many studies. Shirokov [9] determined all two-dimensional pseudo-Riemannian manifolds with common geodesics. The work of Shirokov laid the foundation for systematic research into geodesically corresponding spaces with non-definite metrics. Petrov [8] used the Shirokov's ideas to give a classification of geodesically corresponding pseudo-Riemannian spaces, and his student Golikov [4] determined all four-dimensional Lorentz Spaces with corresponding geodesics. The classification of n-dimensional, geodesically corresponding, Lorentz spaces was completed by Kruchkovich [5]. Aminova [2] studied solving the classic geometrical problem of determining the pseudo-Riemannian metrics that have corresponding geodesics.

The objective of study in this paper is to establish the parametric representation of surface $\boldsymbol{\varphi}(u,v)$ for a given curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u)$. By utilizing the Frenet trihedron frame from differential geometry, we derive the necessary and sufficient conditions for the correct parametric representation of the surface $\boldsymbol{\varphi}(u,v)$ when the parameter u is the arc-length of the curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u)$, and find the necessary constraints on the coefficients of vectors of the frame so that both the geodesic and isoparametric requirements are met. Thus, we define the family of spacelike and timelike surfaces with common geodesic. Also we show with helps of given examples that the member, having any desired property, can be obtained by choosing the appropriate coefficients.

2. Preliminaries

Let us consider Minkowski 3-space $IR^3_1 = [IR^3, (+,+,-)]$ and let the Lorentzian inner product of $\boldsymbol{X} = (x_1, x_2, x_3)$ and $\mathbf{Y} = (y_1, y_2, y_3) \in IR^3_1$ be

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = x_1y_1 + x_2y_2 - x_3y_3.$$

A vector $\boldsymbol{X} \in IR^3_1$ is called a *spacelike* vector when $\langle \boldsymbol{X}, \boldsymbol{X} \rangle > 0$ or $\boldsymbol{X} = 0$. It is called *timelike* and *null* (*lightlike*) *vector* in case of $\langle \boldsymbol{X}, \boldsymbol{X} \rangle < 0$, and $\langle \boldsymbol{X}, \boldsymbol{X} \rangle = 0$ for $\boldsymbol{X} \neq 0$, respectively, [7].

The vector product of vectors $\boldsymbol{X} = (x_1, x_2, x_3)$ and $\boldsymbol{Y} = (y_1, y_2, y_3)$ in IR^3_1 is defined by [1]

$$\mathbf{X} \times \mathbf{Y} = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1). \tag{2.1}$$

Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u)$ be a unit speed curve in IR^3_1; by $\kappa(u)$ and $\tau(u)$ we denote the natural curvature and torsion of $\boldsymbol{\alpha}(u)$, respectively. Consider the Frenet frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ associated with curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u)$ such that $\boldsymbol{e}_1 = \boldsymbol{e}_1(u)$, $\boldsymbol{e}_2 = \boldsymbol{e}_2(u)$ and $\boldsymbol{e}_3 = \boldsymbol{e}_3(u)$ are the unit tangent, the principal normal and the binormal vector fields, respectively. If $\boldsymbol{\alpha} = \boldsymbol{\alpha}(u)$ is a spacelike curve, then the structural equations (or Frenet formulas) of this frame are given as

$$\boldsymbol{e}_1'(u) = \kappa(u)\boldsymbol{e}_2, \quad \boldsymbol{e}_2'(u) = \varepsilon\kappa(u)\boldsymbol{e}_1 + \tau(u)\boldsymbol{e}_3, \quad \boldsymbol{e}_3'(u) = \tau(u)\boldsymbol{e}_2, \tag{2.2}$$

where $\varepsilon = \begin{cases} -1, & \boldsymbol{e}_3 \text{ is timelike}, \\ 1, & \boldsymbol{e}_3 \text{ is spacelike}. \end{cases}$

If $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ is a timelike curve, then above equations are given as ([10])

$$\boldsymbol{e}_1'(u) = \kappa(u)\boldsymbol{e}_2, \quad \boldsymbol{e}_2'(u) = \kappa(u)\boldsymbol{e}_1 - \tau(u)\boldsymbol{e}_3, \quad \boldsymbol{e}_3'(u) = \tau(u)\boldsymbol{e}_2. \tag{2.3}$$

A surface in IR^3_1 is called a *timelike surface* if the induced metric on the surface is a Lorentz metric and is called a *spacelike surface* if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector, [3].

A curve on a surface is geodesic if and only if the normal vector to the curve is everywhere parallel to the local normal vector of the surface, [6].