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Approximations of fractional integrals and Caputo fractional derivatives

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Abstract

In this paper we propose two algorithms for numerical fractional integration and Caputo fractional differentiation. We present a modification of trapezoidal rule that is used to approximate finite integrals, the new modification extends the application of the rule to approximate integrals of arbitrary order $\alpha > 0$. We then, using the new modification derive an algorithm to approximate fractional derivatives of arbitrary order $\alpha > 0$, where the fractional derivative based on Caputo definition, for a given function by a weighted sum of function and its ordinary derivatives values at specified points. The study is conducted through illustrative examples and error analysis.

Keywords: Fractional integral; Caputo fractional derivative; Modified trapezoidal rule; Caputo fractional derivative rule

1. Introduction

1.1. Trapezoidal rule

Numerical integration is a primary tool used by scientists and engineers to obtain approximate answers for definite integrals that can not be solved analytically. Several methods are used to approximate the definite integral of a given function by a weighted sum of function values at specified points. Trapezoidal rule is based on dividing the area between the curve of $f(x)$ and the x-axis into strips and interpolating the function $f(x)$ by a sequence of straight lines.

Trapezoidal rule. Suppose that the interval $[a,b]$ is subdivided into M subintervals $[x_k, x_{k+1}]$ of equal width $h = (b-a)/M$ by using the nodes $x_k = a + kh$, for $k = 0, 1, \ldots, M$. The composite trapezoidal rule for the function $f(x)$ over $[a,b]$ is defined as [1,4]

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$$T(f,h) = \frac{h}{2}\sum_{k=1}^{M}(f(x_{k-1}) + f(x_k)) \tag{1.1}$$

$$= \frac{h}{2}(f(a) + f(b)) + h\sum_{k=1}^{M-1} f(x_k). \tag{1.2}$$

This is an approximation to the integral of $f(x)$ over $[a,b]$, and we write

$$\int_a^b f(x)\,dx \approx T(f,h). \tag{1.3}$$

Trapezoidal rule; *error analysis.* If $f(x) \in \mathbf{C}^2\,[a,b]$, then there is a value c with $a < c < b$ so that the error term $E(f,h)$ has the form

$$E(f,h) = \frac{-(b-a)f^{(2)}(c)h^2}{12} = \mathbf{O}(h^2), \tag{1.4}$$

where

$$E(f,h) = \int_a^b f(x)\,dx - T(f,h). \tag{1.5}$$

1.2. Definitions

Now we will introduce the following definitions and properties of fractional integral and Caputo fractional derivative.

Fractional integral. According to Riemann–Liouville approach to fractional calculus, the fractional integral of order $\alpha > 0$ is defined as [2]

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1} f(\tau)\,d\tau, \quad x > 0. \tag{1.6}$$

Details and properties of the operator J^α can be found in [9,11,12], we mention the following:

For $\alpha, \beta > 0$, $x > 0$ and $\gamma > -1$, we have

$$J^\alpha J^\beta = J^{\alpha+\beta}, \tag{1.7}$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} x^{\gamma+\alpha}, \tag{1.8}$$

$$J^\alpha e^{ax} = x^\alpha \sum_{k=0}^{\infty} \frac{(ax)^k}{\Gamma(\alpha+k+1)}, \tag{1.9}$$

$$J^\alpha \cos(ax) = x^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k (ax)^{2k}}{\Gamma(\alpha+2k+1)}, \tag{1.10}$$

$$J^\alpha \sin(ax) = x^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k (ax)^{2k+1}}{\Gamma(\alpha+2k+2)}. \tag{1.11}$$

Caputo fractional derivative. Let m be the smallest integer that exceeds α, then Caputo fractional derivative of order $\alpha > 0$ is defined as [10]

$$D_*^\alpha f(x) = J^{(m-\alpha)}[f^{(m)}(x)], \tag{1.12}$$

namely

$$D_*^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\left[\int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}}\,d\tau\right], & m-1 < \alpha < m, \\ \frac{d^m}{dx^m} f(x), & \alpha = m. \end{cases} \tag{1.13}$$