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ELSEVIER Applied Mathematics and Computation 175 (2006) 472–485
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

# A two-step algorithm for solving singular linear systems with index one

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**Abstract**

In this paper, a two-step algorithm for solving singular linear systems is presented. We compare this algorithm with a DGCR type algorithm [A. Sidi, A unified approach to Krylov subspace methods for the Drazin-inverse solution of singular non-symmetric linear systems, Linear Algebra Appl. 298 (1999) 99–113] by numerical experiments. An error analysis is given.


*Keywords:* Iterative methods; DGMRES; Two-step algorithm; Singular linear system; Drazin inverse; Index

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[2] Supported by National Natural Science Foundation of China under grant 10471027 and Shanghai Education Committee.

0096-3003/$ - see front matter 
doi:10.1016/j.amc.2005.07.027

## 1. Introduction

Large systems of singular linear equations [6,21]

$$Ax = b, \tag{1}$$

where $A \in C^{N\times N}$ is a singular matrix and $\text{ind}(A) \geqslant 1$, $\text{ind}(A)$ is the index of $A$ that is the size of the largest Jordan block corresponding to the zero eigenvalue of $A$. These systems arise in many different scientific applications. Notably, partial differential equations discretized with finite difference or finite element methods yield systems of singular equations. Large singular linear systems can be solved with either sparse decompositions techniques or with iterative methods. For consistent singular linear systems, these two approaches can be also combined into a method that uses approximate decomposition preconditioning for an iterative method. However, we cannot use preconditioned iterative method for inconsistent singular linear systems.

It is well known that Krylov subspace methods are powerful when they are applied to solve sparse linear system. The Krylov methods concerning with computing singular linear systems have been developed in several papers. First, the method of Conjugate Gradients (CG) can be applied when $A$ is Hermitian positive semidefinite and (1) is consistent, see [10]. It is shown in [14] that the method by Arnoldi [1] and the method of Generalized Conjugate Residuals (GCR) by Eisenstat and Elman [5] and the method of Lanczos [12] can be applied to non-Hermitian but consistent singular systems when $\text{ind}(A)$ is unit and error bounds are also given. Moreover, a type of more general method is provided by Sidi [14–16] and [19,20]. These methods are useful for solving a singular linear system without any limitation. However, some problems arise when these methods are used to compute some ill-conditioned problems, for example, the perturbed steady incompressible Navier–Stokes equations with smaller 'viscosity' parameter (the inverse of the Reynolds number). Thus, preconditioning becomes necessary. There are some preconditioned methods [1,10,14,5,12], but they usually have the limitation that the system (1) must be consistent. However, these type of algorithm cannot be preconditioned since it is possible for (1) to be inconsistent. Obviously, a preconditioned method which is more powerful for the consistent or inconsistent singular linear systems is launching.

The purpose of this paper is to present a preconditioned Krylov subspace method for the Drazin-inverse solution of (1) with unit index. There is no limitation for the singular linear system (1) but the index. The plan of this paper is as follows. In Section 2 we derive the two-step algorithm method. Moreover, In Section 3 we present the saddle point problem. In Section 4, a kind of actual implementation of the two-step algorithm, two-step preconditioned GMRES method, is presented. In Section 5, comparison our two-step preconditioned GMRES method with DGMRES method which is the most stable and efficient in all the projection method through numerical experiment. Furthermore, in