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Church's thesis meets the N-body problem ☆

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Abstract

"Church's thesis" is at the foundation of computer science. We point out that with any particular set of physical laws, Church's thesis need not merely be postulated, in fact it may be decidable. Trying to do so is valuable. In Newton's laws of physics with point masses, we outline a proof that Church's thesis is false; physics is unsimulable. But with certain more realistic laws of motion, incorporating some relativistic effects, the extended Church's thesis is true. Along the way we prove a useful theorem: a wide class of ordinary differential equations may be integrated with "polynomial slowdown". Warning: we cannot give careful definitions and caveats in this abstract–you must read the full text—and interpreting our results is not trivial.

Keywords: Newtonian N-body problem; Church's thesis; Computability; Numerical methods for ordinary differential equations

1. Introduction. Our results and their interpretation

"*Church's thesis*", or the "Church-Turing thesis" [46,32], states that the *set of things commonly understood to be computation is identical with the set of tasks that can be carried out by a Turing machine.*

At first, Church's thesis seems merely to be a definition of the word "computation" and thus content-free. Indeed, it does have some of a character somewhere between that of a definition and an assertion, which is why it is always stated in an intentionally slightly vague way.

However, it can also be interpreted as a profound claim about the physical laws of our universe, i.e.: *any physical system that purports to be a "computer" is not capable of any computational task that a Turing machine is incapable of.*

Definition 1. If computer A will always complete a task whose input is L bits long in time $T(L)$, and computer B always does the same task in time $\leqslant P(T(L), L)$ where P is a polynomial, then B is said to have "polynomial slowdown" relative to A.

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The "extended" Church thesis states that a Turing machine can do anything any other kind of (physically realizable) computer can do, with at most polynomial slowdown.[1]

Church's thesis lies at the heart of theoretical computer science and physics; if it were false, much of the life's work of most computer scientists and theoretical physicists would become worthless, or at least, worth less.

So an important question is now to try to formulate certain sets of physical laws and to try to determine whether Church's thesis or the extended Church's thesis would be valid in a universe with those physical laws. A way to prove the (extended) Church's thesis is to construct an (efficient) algorithm for simulating any physical system. A way to disprove Church's thesis is to show how to use the laws of physics to construct a "computer" that can do something that Turing machines cannot do.

Let N be a fixed whole number. The "Newtonian N-body problem" is to describe the motion of N point masses whose initial locations and velocities are given, assuming that Newton's law of gravity $F = Gm_1m_2/r^2$ and acceleration $F = ma$ hold. We will sketch a proof (it is based on Gerver's proof of the "Painlevé conjecture" in the plane) of Theorem 4 that an *uncountably* infinite number of *topologically distinct* trajectories are possible in 1 s, among the planar N-body problems with fixed masses and whose initial locations lie within certain disjoint balls and whose velocities are bounded. Meanwhile, of course, Turing machines can only experience a finite number of possible histories in a finite time. As a consequence, it is impossible for a Turing machine to compute a correct qualitative description of the motion that N bodies can make in 1 s. (Here, of course, "1 s" could be replaced by any finite interval of time.)

This result can be interpreted as "the (unextended) Church thesis would not be valid in a universe with point masses and Newton's laws of motion". This interpretation of our result is muddled by the fact that Newtonian physics involves real numbers specified infinitely precisely. However, the topological distinctness statement that we prove is discrete. Let us be clearer.

1.1. First way to interpret Theorem 4: N-bodies are unsimulatable

A Turing machine could be given N real numbers as input by simply providing it with N infinite tapes each containing the binary representation of a number. This is in fact the realistic model of input if the task is N-body simulation. Such a Turing machine *could*, in infinite computational time, calculate the topology of the trajectories of N bodies up to the point in time (if any) where a singularity occurs. In fact, if the real numbers specifying the initial locations and momenta of the bodies were "usual", the Turing machine would in fact succeed in completing this calculation (for any particular desired amount of simulated time) in *finite* time. But, if the real numbers happened to be "unusual", in fact, if they happened to correspond to one of our examples whose trajectory topologies achieved singularity and infinite complexity in 1 s, then in finite computer time it could only partially describe the topology of the trajectory and would have to keep reading more input bits forever. (Probably our examples truly are unusual, in the sense that[2] they are measure zero in the space of real number tuples. However, we have not proven this. See Section 4.4.) Thus, this way of looking at it makes it apparent that the N-body system really can do something a Turing machine cannot do.

[1] In the event that the physical system is not deterministic, then the Turing machine has to be given a random bit generator, and the criteria for completion of a computational task would have to become statistical. We will not concern ourselves with this in the present paper. Also, it is naturally essential, in order to give the physical universe any chance in the competition with a Turing machine having an infinite tape, that the laws of physics consider the universe to have infinite extent.

[2] In which case, it could be argued that these unusual examples correspond to a zero volume set in phase space. Compressing the phase space volume of a physical system exponentially small (e^{-n}) seems to require (by the second law of thermodynamics) increasing the entropy of the outside universe (outside this system, that is) by at least nK_B, which, assuming the outside universe was at temperature T, would require dissipating energy $\geqslant nk_BT$. Thus, setting up the initial conditions with infinite accuracy would require infinite energy expenditure, a conclusion which certainly must be taken into account when evaluating my claim to have "disproved Church's thesis". I still claim there is a sense in which N point masses with Newton's laws are more powerful than a Turing machine, but the present footnote exhibits a sense in which that is not the case.