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Existence of fractional integral equation with hysteresis

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Abstract

The existence of solutions to an integral equation of fractional order with hysteresis has been established. A fixed point theorem due to Schauder is the main tool in carrying out our proof.

Keywords: Fractional integral; Hysteresis; Schauder's fixed point theorem

1. Introduction

The first serious attempt to give a logical definition of a fractional derivative is due to Liouville, see [12] and references therein. Now, the fractional calculus topic is enjoying growing interest among scientists and engineers, see [7,9–12,15,16]. The fractional order integral (Riemann–Liouville) operator is a singular integral, for example see [10] and references therein. On the other hand, the mathematical study of hysteresis phenomena is relatively new. The theory of hysteresis operators developed in the past thirty years or so. It was only in the seventies that Krasnosel'skii and Pokrovskii [13]. There are various ways how hysteretic behavior of a system can be related to an integral equation. One particular setting, which has been studied by many authors, is using a convolution integral to describe the memory of a given system. The memory is characterized by the convolution kernel and thus the evolution depends on all past values of the state; typically, as one goes back in time, the influence of the past values of the present evolution decreases. There are, however, several hysteretic phenomena which cannot be treated by this way; in particular, it cannot be used to describe a hysteretic system whose hysteresis loops do not depend on the speed with which they are traversed. This property is called rate independence and is inherently nonlinear, for instance, see [1,13,14,17]. In [2–6], the author discusses systems where a Urysohn–Volterra integral equation is coupled to a rate independent hysteretic process.

In this paper we study the following nonlinear integral equation with hysteresis:

$$y(t) = f(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(s, y(s), \mathscr{W}[S[y]](s))}{(t-s)^{1-\alpha}}\,\mathrm{d}\tau, \quad 0 \leqslant t \leqslant T. \tag{1.1}$$

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Here $\mathscr{W}$ and S denote a hysteresis operator and a superposition operator of the form $S[y](t) = k(y(t))$, respectively.

Definition 1. Let $g \in L_1(a,b)$, $0 \leqslant a < b < \infty$, and let $\alpha > 0$ be a real number. The fractional integral of order α is defined by (see [15,16])

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} \mathrm{d}s.$$

Rewrite Eq. (1.1) in the form

$$y(t) = f(t) + I^\alpha u(t, y(t), \mathscr{W}[S[y]](t)), \quad 0 \leqslant t \leqslant T, \tag{1.2}$$

where I^α is the standard Riemann–Liouville fractional integral. There are several works concerning the analytic solutions of Eq. (1.2) without hysteresis and its related types, for example see [9–12]. None of the authors, however, has provided analytic or numerical solutions of Eq. (1.2).

In this paper, we prove an existence theorem of solutions to Eq. (1.2).

2. Preliminaries

In this section, we state some results needed in the proof of our main theorem.

Let $C([0,T];\mathbb{R}^n)$ be the Banach space of all continuous functions $\phi : [0,T] \to \mathbb{R}^n$ endowed with the sup-norm $\|\phi\| = \sup_{0 \leqslant \theta \leqslant T} \|\phi(\theta)\|_{\mathbb{R}^n}$.

By a *hysteresis operator* we mean a Volterra operator which possesses the additional property of rate independence.

Definition 2 (*Rate independent functionals*). A functional $\mathscr{H} : C([0,T];\mathbb{R}^n) \to \mathbb{R}$; is called rate independent if and only if $\mathscr{H}[u \circ \psi] = \mathscr{H}[u]$ holds for all $u \in C([0,T];\mathbb{R}^n)$ and all admissible time transformations, i.e., continuous increasing functions $\psi : [0,T] \to [0,T]$ satisfying $\psi(0) = 0$ and $\psi(T) = T$.

Definition 3 (*Volterra-operator*). Let X be a Banach space. An operator $F : C([0,T];X) \to C([0,T])$ is called a Volterra-operator if for all $s \in [0,T]$ and for all $u, v \in C([0,T];X)$ with $u(\sigma) = v(\sigma)$ for all $\sigma \in [0,s]$ implies $(Fu)(\sigma) = (Fv)(\sigma)$ for all $\sigma \in [0,s]$.

Lemma 1 [6]. *Let $F : C([0,T];\mathbb{R}^n) \to C([0,T])$ be a Volterra-operator. Assume that F is Lipschitz continuous on every bounded subset of $C([0,T];\mathbb{R}^n)$. Then for every $C > 0$ there exists $L > 0$ such that*

$$|(Fy_2)(s) - (Fy_1)(s)| \leqslant L \sup_{0 \leqslant \sigma \leqslant s} \|y_2(\sigma) - y_1(\sigma)\|_{\mathbb{R}^n}, \tag{2.1}$$

holds for all $0 \leqslant s \leqslant T$ and all $y_i \in C([0,T];\mathbb{R}^n)$ with $\|y_i\| \leqslant C$, $i = 1,2$.

Our existence theorem is based on the following fixed point theorem, [8].

Lemma 2 [Schauder's fixed point theorem]. *Let C be a convex subset of a Banach space E and $\mathscr{F}$ be a completely continuous mapping of C into C. Then $\mathscr{F}$ has at least one fixed point in C.*

Remark 1. By definition hysteresis operators possess the Volterra property. This is actually what is needed here; the rate-independence itself does not play any role.

3. Main theorem

Assumptions 1. Let T, b and M be positive numbers. Take D_1 be the set of vectors $y \in \mathbb{R}^n$ for which there exists an $s \in [0,T]$ such that $\|y - f(s)\| \leqslant b$ and D_2 be the set of all $w \in \mathbb{R}$ such that $|w| \leqslant b$.

Furthermore, let $u : D \to \mathbb{R}^n$, $D \equiv [0,T] \times D_1 \times D_2$, be a continuous function with the following property

$$\sup_{\zeta \in D} \|u(\zeta)\|_{\mathbb{R}^n} \leqslant M.$$