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Nontrivial solution of third-order nonlinear eigenvalue problems ☆

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Abstract

In this paper, we study the existence and uniqueness of nontrivial solution for the following third-order eigenvalue problems (TEP):

$$\begin{cases} u''' = \lambda f(t,u,u'), \quad 0 < t < 1, \\ u(0) = u'(\eta) = u''(0) = 0, \end{cases}$$

where $\lambda > 0$ is a parameter, $\frac{1}{2} \leqslant \eta < 1$ is a constant, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $\mathbb{R} = (-\infty, +\infty)$. Without any monotone-type and nonnegative assumption, we obtain serval sufficient conditions of the existence and uniqueness of nontrivial solution of TEP when λ in some interval. Our approach is based on Leray–Schauder nonlinear alternative.

Keywords: Nontrivial solution; Eigenvalue problem; Fixed point; Leray–Schauder nonlinear alternative

1. Introduction

In this paper, we are concern with the existence and uniqueness of nontrivial solution for the following third-order eigenvalue problems (TEP):

$$\begin{cases} u''' = \lambda f(t,u,u'), \quad 0 < t < 1, \\ u(0) = u'(\eta) = u''(0) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\frac{1}{2} \leqslant \eta < 1$ is a constant, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $\mathbb{R} = (-\infty, +\infty)$.

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The third-order boundary value problem arises in the study of draining and coating flows. In the past few years, by applying the Leray–Schauder continuation theorem, the coincidence degree theory, the Krasnosel'skii fixed point theorem, many authors have studied certain boundary value problems for nonlinear third-order ordinary differential equations, for details, see [1–4,6,7] and their references therein. Recently, the existence and multiplicity of positive solutions for the TEP have been studied extensively, for example, Anderson and Avery [5] used the Leggett–Williams fixed point theorem to obtain three solutions for the third-order boundary value problem:

$$\begin{cases} u''' = f(u), & 0 < t < 1, \\ u(0) = u'(t_2) = u''(1) = 0, & \frac{1}{2} \leqslant t_2 < 1. \end{cases}$$

But in the existing literature, few people considered the case where nonlinear term is involved explicitly with the first-order derivative. The aim of this paper is to establish some results on existence and uniqueness of nontrivial solution for the TEP (1.1). Our results are new and different from those of [1–5]. Particularly, we do not need any monotonicity and nonnegative assumptions on f, which was essential for the technique used in [1–5].

2. Preliminaries and lemmas

Let $X = C[0,1]$ be endowed with the ordering $x \leqslant y$ if $x(t) \leqslant y(t)$ for all $t \in [0,1]$, and $\|u\| = \max_{t\in[0,1]}|u(t)|$ is defined as usual by maximum norm. Now we introduce the norm in $Y = C^1[0,1]$ by

$$\|u\|_1 = \|u\| + \|u'\| = \max_{t\in[0,1]}|u(t)| + \max_{t\in[0,1]}|u'(t)|.$$

Clearly, it follows that $(Y, \|\cdot\|_1)$ is a Banach space.

Lemma 2.1. *Let $y(t) \in X$, then the BVP*:

$$\begin{cases} u''' - y(t) = 0, & 0 < t < 1, \\ u(0) = u'(\eta) = u''(0) = 0, \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} t\min\{\eta, s\} - \frac{1}{2}t^2, & 0 \leqslant t \leqslant s \leqslant 1, \\ t\min\{\eta, s\} + \frac{1}{2}s^2 - ts, & 0 \leqslant s \leqslant t \leqslant 1 \end{cases}$$

is Green's function of the BVP

$$\begin{cases} u''' = 0, & 0 < t < 1, \\ u(0) = u'(\eta) = u''(0) = 0. \end{cases}$$

Proof. The proof of this lemma is easy, and we omit it. □

Remark 2.1. It is obvious that the Green's function $G(t,s)$ is continuous and $G(t,s) \geqslant 0$ for any $0 \leqslant t, s \leqslant 1$. In addition, we also have

$$\max_{0\leqslant t,s\leqslant 1} G(t,s) \leqslant \frac{1}{2}.$$