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Solvability of Urysohn and Urysohn–Volterra equations with hysteresis in weighted spaces

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Abstract

The existence of solutions to nonlinear integral equations of the second kind with hysteresis, of Urysohn–Volterra and Urysohn types has been established. We develop the solvability theory of Urysohn–Volterra equation with hysteresis in weighted spaces proposed by the author [M.A. Darwish, On solvability of Urysohn–Volterra equations with hysteresis in weighted spaces, J. Integral Equat. Appl. 14 (2) (2002) 151–163].

Keywords: Urysohn; Volterra; Existence and uniqueness; Weighted spaces; Hysteresis operator

1. Introduction

There are various ways how hysteretic behavior of a system can be related to an integral equation. One particular setting, which has been studied by many authors, is using a convolution integral to describe the memory of a given system. The memory is characterized by the convolution kernel and thus the evolution depends on all past values of the state; typically, as one goes back in time, the influence of the past values of the present evolution decreases. There are, however, several hysteretic phenomena which cannot be treated by this way; in particular, it cannot be used to describe a hysteretic system whose hysteresis loops do not depend on the speed with which they are traversed. This property is called rate independence and is inherently nonlinear. In [2–6], we discuss systems where a Urysohn–Volterra integral equation is coupled to a rate independent hysteretic process. For more information about hysteresis, for instance, see [1,8,12].

In this paper we study nonlinear integral equations of the second kind with hysteresis, namely

$$y(t) = f(t) + \int_{-\infty}^{t} F(t,s,y(s),\mathscr{W}[S[y]](s))\mathrm{d}s, \tag{1.1}$$

$$y(t) = f(t) + \int_{-\infty}^{\infty} F(t,s,y(s),\mathscr{W}[S[y]](s))\mathrm{d}s. \tag{1.2}$$

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Here, $\mathscr{W}$ and S denote a hysteresis operator and a superposition operator of the form $S[y](t) = k(y(t))$, respectively. More precisely, we assume f and F to be given n-vector valued functions, while y is the unknown n-vector function. Eqs. (1.1) and (1.2) are known as Urysohn–Volterra and Urysohn integral equations, respectively [3].

Eq. (1.1) is history-dependent so, in general, this problem requires that one give an initial condition on $(-\infty, 0]$ and it may be treated with the techniques of standard Urysohn–Volterra equations with hysteresis, see Darwish [4–6]. Therefore, the nonuniqueness of solutions of Eq. (1.1) is an intrinsic feature which occur even in the case of linear integral equations without hysteresis. For example, the equation

$$y(t) = \mathrm{e}^{-t} + \frac{1}{2}\int_{-\infty}^{t} \mathrm{e}^{-2(t-s)} y(s)\mathrm{d}s \tag{1.3}$$

has the solution set $\{y(t) = 2\mathrm{e}^{-t} + c\mathrm{e}^{-3/2t} : c \text{ is a constant}\}$ (cf. [10]). However, in [9,11] it was observed that uniqueness of Eqs. (1.1) and (1.2) without hysteresis occurs in some kind of weighted spaces. Recently, Darwish [3] observed that uniqueness of Eq. (1.1) occurs in some weighted spaces.

The main object of this paper is to show the existence of solutions of Eq. (1.1) and develop the solvability theory of Urysohn–Volterra equation with hysteresis in weighted spaces proposed by Darwish [3]. More precisely, in Section 3 we relax the restrictions imposed on the function F in [3] and obtain the existence and uniqueness of solutions of Eq. (1.1). Also, our method is applicable to an Urysohn equation with hysteresis.

2. Preliminaries

In this section, we state some results needed in the proof of our main theorems.

Let $I \subset \mathbb{R}$ and consider a weight function $w : I \to \mathbb{R}_+$ be continuous and nondecreasing, $\mathbb{R}_+ = (0, +\infty)$. Define $C_w \equiv C_w(I; \mathbb{R}^n) := \{\phi | \phi : I \to \mathbb{R}^n \text{ continuous}\}$ with the following norm:

$$\|\phi\|_w = \sup_{t\in I} \frac{\|\phi(t)\|_{\mathbb{R}^n}}{w(t)}, \quad \forall \phi \in C_w$$

to be the underlying space for our problem. Then the spaces $(C_w, \|\cdot\|_w)$ is a Banach space.

Definition 1 (*Rate independent functionals*). A functional $\mathscr{H} : C([0, T]; \mathbb{R}^n) \to \mathbb{R}^n$; is called rate independent if and only if $\mathscr{H}[u \circ \psi] = \mathscr{H}[u]$ holds for all $u \in C([0, T]; \mathbb{R}^n)$ and all admissible time transformations, i.e., continuous increasing functions $\psi : [0, T] \to [0, T]$ satisfying $\psi(0) = 0$ and $\psi(T) = T$.

Definition 2 (*Volterra-operator*). Let X be a Banach space. An operator $F: C([0, T]; X) \to C([0, T])$ is called a Volterra-operator if for all $s \in [0, T]$ and for all $u, v \in C([0, T]; X)$ with $u(\sigma) = v(\sigma)$ for all $\sigma \in [0, s]$ implies $(Fu)(\sigma) = (Fv)(\sigma)$ for all $\sigma \in [0, s]$.

Recall that an operator $\mathscr{W} : C(I; \mathbb{R}^n) \to C(I)$ is hysteresis if it has the Volterra property and the rate independent property. For more information about hysteresis operator, see [1] and the references therein.

Remark 1. By definition hysteresis operators possess the Volterra property. This is actually what is needed here; the rate-independence itself does not play any role.

Lemma 1 [3]. *Let $F : C(I; \mathbb{R}^n) \to C(I)$ be a Volterra-operator. Assume that F is Lipschitz continuous on every bounded subset of $C(I; \mathbb{R}^n)$. Then for every $C > 0$ there exists $L > 0$ such that*

$$|(Fy_2)(s) - (Fy_1)(s)| \leqslant L \sup_{\substack{\tau\in I\\ \tau\leqslant s}} \|y_2(\tau) - y_1(\tau)\|_{\mathbb{R}^n}, \tag{2.1}$$

holds for all $s \in I$ and all $y_i \in C(I; \mathbb{R}^n)$ with $\|y_i\| \leqslant C$, $i = 1, 2$.

3. The unique solvability

Let I be a (bounded or unbounded) closed subinterval of $\mathbb{R}$ and define C_w as above. To facilitate our discussion, let us first state the following assumptions: