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Numerical accuracy of a certain class of iterative methods for solving linear system

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Abstract

One of the most important problem for solving the linear system $Ax = b$, by using the iterative methods, is to use a good stopping criterion and to determine the common significant digits between each corresponding components of computed solution and exact solution. In this paper, for a certain class of iterative methods, we propose a way to determine the number of common significant digits of x_m and x, where x_m and x are computed solution at iteration m and exact solution, respectively. By using the CADNA library which allows us to estimate the round-off error effect on any computed result, we also propose a good stopping criterion which is able to stop the process as soon as a satisfactory informatical solution is obtained. Numerical examples are used to show the good numerical properties.

Keywords: Iterative methods; Significant digits; FOM algorithm; Error propagation; CESTAC method; Stochastic arithmetic; CADNA library

1. Introduction

Consider the linear system

$$Ax = b, \tag{1}$$

where A is a real nonsingular matrix of order N. For solving Eq. (1) by an iterative method, one can use the common strategy for stopping the iterations. For example, for a given tolerance $\epsilon > 0$, we can use the following stopping criterions:

1. $\|x_n - x_{n-1}\| < \epsilon$,
2. $\|b - Ax_n\| < \epsilon$,

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where $\|\cdot\|$ is a vector norm and $\{x_n\}$ is a sequence of vectors such that

$$\lim_{n\to\infty} x_n = x = A^{-1}b. \tag{2}$$

If one of the above stopping criterions is used then the number of significants digits that are common to corresponding entries of x_n and x cannot be specified. Another problem is to choose the value ϵ. When ϵ is chosen too large, then the iterative process is stopped too soon, and consequently the approximate solution has a poor accuracy. On the contrary, when ϵ is chosen too small, it is possible, due to the numerical instabilities, that many useless iterations are performed without improving the accuracy of the solution. The aim of this paper is to solve these problems.

This paper is organized as follows. In Section 2, two theorems are proved in order to derive a lower bound for the common significant digits of each corresponding components of x_m and x. In Section 3, a brief description of stochastic round-off analysis, the CESTAC method and the CADNA software is described. In Section 4, by using the theorems of Section 2 and the CADNA library which allows us to estimate the round-off error effect on any computed result, we propose a good stopping criterion which is able to stop the process as soon as a satisfactory informatical solution is obtained and to estimate the number of significant digits of its components. In Section 4, some numerical results are also given to show the good numerical properties.

2. Theoretical results

2.1. Preliminaries

As in [7], we recall the following definition:

Definition 1. Let p and q be two real numbers, the number of significant digits that are common to p and q can be defined in $(-\infty, +\infty)$ by

1. for $p \neq q$,

$$C_{p,q} = \log_{10}\left|\frac{p+q}{2(p-q)}\right|.$$

2. $\forall p \in \mathbb{R},\ C_{p,p} = +\infty.$

For two vectors $a, b \in \mathbb{R}^n$ we define the following number:

1. for $a \neq b$

$$C_{a,b} = \log_{10}\frac{\|a+b\|_2}{2\sqrt{n}\|a-b\|_2}. \tag{3}$$

2. $\forall a \in \mathbb{R}^n,\ C_{a,a} = +\infty.$

Now, in the following lemma, we shall develop the important property of the number $C_{a,b}$.

Lemma 1. *If $a = (a_i)$ and $b = (b_i)$ are two vectors in $\mathbb{R}^n$ and*

$$\frac{|a_i + b_i|}{\|a+b\|_\infty} = \alpha_i \times 10^{-\beta_i}, \tag{4}$$

where $0.1 \leqslant \alpha_i < 1$ and $\beta_i \geqslant -1$, for $i = 1, \ldots, n$, then

$$C_{a_i,b_i} \geqslant C_{a,b} - (\beta_i + 1).$$

Proof. By using (3), we can write

$$|a_i - b_i| \leqslant \|a-b\|_2 = \frac{\|a+b\|_2}{2\sqrt{n}} 10^{-C_{a,b}} \leqslant \frac{\|a+b\|_\infty}{2} 10^{-C_{a,b}} \tag{5}$$