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Numerical validation of blow-up solutions of ordinary differential equations

Akitoshi Takayasu [a,*], Kaname Matsue [b], Takiko Sasaki [c], Kazuaki Tanaka [d], Makoto Mizuguchi [d], Shin'ichi Oishi [e]

[a] Faculty of Engineering, Information and Systems, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8573, Japan
[b] The Institute of Statistical Mathematics, 10-3 Midori-cho, Tachikawa, Tokyo 190-8562, Japan
[c] Global Education Center, Waseda University, 1-104 Totsuka-machi, Shinjuku, Tokyo 169-8050, Japan
[d] Graduate School of Fundamental Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan
[e] Department of Applied Mathematics, Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan

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ABSTRACT

This paper focuses on blow-up solutions of ordinary differential equations (ODEs). We present a method for validating blow-up solutions and their blow-up times, which is based on compactifications and the Lyapunov function validation method. The necessary criteria for this construction can be verified using interval arithmetic techniques. Some numerical examples are presented to demonstrate the applicability of our method.

1. Introduction

In this paper, we consider the initial value problem defined by the following ordinary differential equations in $\mathbb{R}^m$ ($m \in \mathbb{N}$):

$$\frac{dy(t)}{dt} = f\left(y(t)\right), \quad y(0) = y_0, \tag{1}$$

where $t \in [0, T)$ with $0 < T \leq \infty$, $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 function, and $y_0 \in \mathbb{R}^m$. Unless otherwise noted, f is assumed to be a polynomial, whose coefficients are real numbers. Our focus in this paper is a class of solutions of (1) called *blow-up solutions*.

* Corresponding author.
E-mail address: takitoshi@risk.tsukuba.ac.jp (A. Takayasu).

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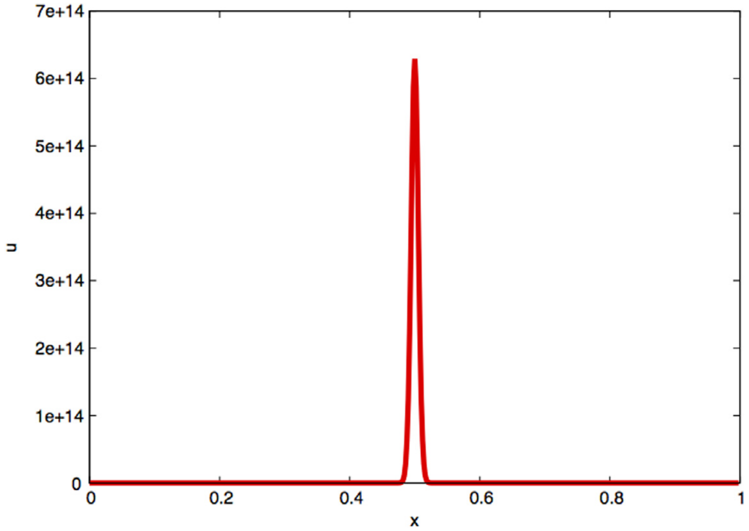

Fig. 1. Numerical blow-up solution of (3) with $p = 3$. The L^∞-norm of solutions becomes larger and larger, and may become infinite within a finite time. We typically regard solutions whose L^∞-norms become sufficiently large within finite times as "blow-up solutions" in a numerical sense.

Definition 1.1. Define $t_{\max} > 0$ as

$$t_{\max} := \sup \left\{ \bar{t} : \text{a solution } y \in C^1([0, \bar{t})) \text{ of (1) exists} \right\}.$$

We say that the solution y of (1) *blows up* if $t_{\max} < \infty$. In such a case, $t_{\max}$ is called the *blow-up time* of (1).

The simplest example of blow-up phenomena can be seen for the following ordinary differential equation (ODE) in $\mathbb{R}^1$:

$$\frac{dy}{dt} = y^2, \quad y(0) = y_0. \tag{2}$$

When $y_0 > 0$, the exact solution of (2) is $y(t) = (y_0^{-1} - t)^{-1}$. The value of $y(t)$ becomes infinite as $t \to y_0^{-1} - 0$. That is, $y(t)$ blows up at $t = y_0^{-1}$. Blow-up solutions can also be observed for partial differential equations (PDEs), such as the nonlinear heat equations (e.g., [1,2]) given by

$$u_t = \Delta u + |u|^{p-1}u, \quad p > 1, \tag{3}$$

the nonlinear wave equations (e.g., [3,4]), and the nonlinear Schrödinger equations (e.g., [5]). In the case of PDEs, many researchers have studied blow-up phenomena such as blow-up times, blow-up criteria, the behavior of solutions near blow-up times (e.g., blow-up rate), and the topology or geometry of blow-up sets. Studies of blow-up phenomena can be of importance both mathematically and physically. For example, in the case of the nonlinear heat equation (3), blow-up solutions describe the combustion of solid fuels [6]. Similarly, the blow-up time corresponds to the time when the fuel ignites. Blow-up phenomena associated with (3) thus describe the process of combustion.

The numerical analysis of blow-up solutions, such as of nonlinear heat and reaction–diffusion equations [7–11], of nonlinear wave equations [12,13], and of nonlinear Schrödinger equations [14,15], has also been studied. However, in almost all numerical studies concerning blow-up solutions, "blow-up solutions" have been only computed approximately. For example, typical numerical computations of blow-up solutions begin by setting an appropriately large number M, say 10^6. Then, one numerically solves the differential equations, and regards computed solutions whose supremum norms are larger than M as blow-up solutions (Fig. 1). However, this criterion provides us with no proof that these computed blow-up solutions are *rigorous* blow-up solutions. In other words, it is possible that "numerical blow-up solutions" just describe extremely large but *bounded* solutions. For example, consider (2) again, and the perturbed equation

$$\frac{dy}{dt} = y^2 - \epsilon y^3, \quad y(0) = y_0 > 0,$$

where $\epsilon > 0$ is a sufficiently small parameter. One can easily see that the solution tends to $y = 1/\epsilon$ as $t \to \infty$. Obviously, this solution is not a blow-up solution, while the dominant behavior of this solution resembles that of $dy/dt = y^2$. In such a case for a general system, it is not easy to judge whether a computed solution is truly a blow-up solution. Therefore, an exact criterion for blow-up solutions is necessary to concretely obtain rigorous blow-up solutions.

The blow-up time $t_{\max}$ is one of the key considerations for blow-up solutions. Some specific solutions, such as self-similar solutions, can be described via transformations involving $t_{\max}$ (see, e.g., [16]), in which case we assume that $t_{\max}$ is known in advance. However, the detection of $t_{\max}$ itself is not easy, because $t_{\max}$, in general, depends on an initial condition (as