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A priori error estimates of the DtN-FEM for the transmission problem in acoustics

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ABSTRACT

This paper is concerned with a variational approach solving the two-dimensional acoustic transmission problems. The original problem is reduced to an equivalent nonlocal boundary value problem by introducing an exact Dirichlet-to-Neumann (DtN) mapping in terms of Fourier expansion series on an artificial boundary. Uniqueness and existence of solutions in appropriate Sobolev spaces are established for the corresponding variational problem and its modification due to the truncation of DtN mapping. A priori error estimates containing the effects of both element meshsize and truncation order of series for the finite element approximation are derived. Numerical experiments are also presented to illustrate the efficiency and accuracy of the numerical scheme.

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1. Introduction

Theoretical analysis and numerical approximation for the scattering problems of time-harmonic acoustic waves by an impenetrable or penetrable obstacle have been studied [1–10] for many years. These problems suffer from many mathematical and computational challenges such as the oscillating character of solutions [11–14] and the unbounded domain to be considered. In this paper, we are interested in the numerical solutions for the two dimensional acoustic transmission problem with isotropic medium.

It is known that numerical difficulties resulting from the fact that the scattered wave propagates in an unbounded region can be circumvented by several strategies. The boundary integral equation methods based on Green's formula and potential theory allow us to reduce the original problem to systems of boundary integral equations, see [1,5,7] for acoustic transmission problems. Another technique is to decompose the unbounded domain first by introducing an artificial boundary containing the obstacle inside. Then appropriate methods can be employed to solve the exterior scattering problem outside the artificial boundary while field equation solvers are used for the solution on the bounded domain inside the artificial boundary. There are several techniques [15–18,9] to realize such a coupling procedure based on the above domain decomposition scheme. One of them is to applying boundary integral equation methods to solve the exterior scattering problem and to using finite element methods (FEM) to solve the interior one, and this process leads to a coupling of FEM and boundary element methods (BEM) [16,17]. On the other hand, one can enforce some artificial boundary condition on the auxiliary boundary [19]. In particular, we can derive a Dirichlet-to-Neumann (DtN) mapping on the artificial boundary curve to obtain a reduced nonlocal boundary value problem which could be solved by FEM, and accordingly this strategy yields the so-called the DtN-FEM [20]. There are two main methods for the derivation of such DtN mappings. Representing the DtN mapping

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by boundary integral operators [21,9,22] and Fourier expansion series [15,23,4] leads to another form of coupling of FEM and BEM, and the coupling of FEM and the method of separation of variables, respectively. The Fourier expansion series based DtN mapping can only be defined on a restricted shape (circle or ellipse in two dimensions) of artificial boundaries or their finite perturbations [10,24]. Corresponding to the exact DtN mapping, there are also several types of local boundary conditions [25–27] and the simplest one can be obtained by employing the Sommerfeld condition directly on the artificial boundary. This paper is designed to make contributions to the error analysis of Fourier-series-based DtN-FEM solving the transmission problem in acoustics.

In practical application, the infinite series in DtN mapping should be truncated into a finite one and the corresponding number of remaining terms is called the truncation order. Then the numerical errors of approximated solutions contain both effects of the finite element discretization and the series truncation. This has been considered for exterior Laplace problems [28], exterior Helmholtz problems [8,4] and fluid–solid interaction problems [29]. In this work, we first reduce the original acoustic transmission problem to an equivalent nonlocal boundary value problem by introducing an exact Fourier-series-based DtN mapping defined on a circle which is the artificial boundary enclosing the obstacle. Then by using the properties of the sesquilinear form of the DtN mapping terms analyzed in [4] and the technique in [30] for interior acoustic transmission problem, we show that both the corresponding variational problem of the nonlocal boundary value problem and its modification due to truncation of the DtN mapping satisfy a Gårding's inequality and admit a unique weak solution. Furthermore, these two features allow us to establish the inf–sup condition [31] as a consequence of the general theorem in [32] provided that the finite element space satisfies the approximation property. Starting with the inf–sup condition, we are able to derive an upper bound of numerical errors analogous to the well-known *Céa's* lemma in the positive definite case. Since we are now finding solutions in a special Sobolev space due to the transmission conditions, a prior error estimates measured in $\mathcal{H}^0$-norm and $\mathcal{H}^1$-norm including the effects of both the discretization error and the truncation error are derived by following the techniques in [30] for interior transmission problems different from those in [4] for exterior acoustic scattering problem. Finally, compared to the exact Fourier-series-based DtN mapping, we also simulate one type of local DtN mapping proposed in [27] in numerical experiments.

The paper is organized as follows. In Section 2, we give a brief introduction to the considered acoustic transmission problem. In Section 3, we propose an equivalent nonlocal boundary value problem in a truncated domain via an exact DtN mapping and then present the solvability results of the corresponding variational problem. Section 4 is devoted to the solvability of the modified variational problem due to the truncation of the DtN mapping. We establish a priori error estimates for the Galerkin solution in appropriate Sobolev spaces in Section 5. Finally, Section 6 presents several numerical tests to illustrate the efficiency and accuracy of Fourier-series-based DtN-FEM and we make conclusions in Section 7.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ denote a penetrable obstacle which is a bounded polyhedral domain with boundary Γ, and $\Omega^c = \mathbb{R}^2 \setminus \overline{\Omega}$ be the unbounded exterior domain outside Γ (see Fig. 1). Assume that u^i is an incident wave described by the wave number $k_2 := \omega/c_2$ with frequency ω and sound speed $c_2 > 0$ of the acoustic wave in the background medium Ω^c. In addition, we denote by c_1 the sound speed of the acoustic wave in Ω and $k_1 = \omega/c_1$ the corresponding wave number. Now the classical direct acoustic transmission problem reads: *Given an incident field u^i, find the transmitted field $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and the scattered field $u^s \in C^2(\Omega^c) \cap C^1(\overline{\Omega^c})$ satisfying*

$$\Delta u + k_1^2 u = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$\Delta u^s + k_2^2 u^s = 0 \quad \text{in } \Omega^c, \tag{2.2}$$

$$u - u^s = u^i \quad \text{on } \Gamma, \tag{2.3}$$

$$\frac{\partial u}{\partial n} - \frac{\partial u^s}{\partial n} = \frac{\partial u^i}{\partial n} \quad \text{on } \Gamma, \tag{2.4}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial u^s}{\partial r} - ik_2 u^s \right) = 0, \quad r = |x|, \tag{2.5}$$

which holds uniformly with respect to $\hat{x} = x/|x| \in \mathbb{S} := \{\hat{\theta} \in \mathbb{R}^2 : |\hat{\theta}| = 1\}$. Here, $i = \sqrt{-1},\ x = (x_1, x_2) \in \mathbb{R}^2$.

The uniqueness of the problem (2.1)–(2.5), under some assumptions on the wave numbers k_1 and k_2, is given in the following lemma [7].

Lemma 2.1. *Let $k_1 \neq 0$, $k_2 \neq 0$ be such that $\mathrm{Im}(k_2) \geq 0$ and $\mathrm{Im}(k_1^2 \overline{k_2}) \geq 0$. Then the classical transmission problem (2.1)–(2.5) has at most one solution.*

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