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Asymptotic results for a Markov-modulated risk process with stochastic investment

Lewis Ramsden *, Apostolos D. Papaioannou

Institute for Financial and Actuarial Mathematics, Department of Mathematical Sciences, University of Liverpool, Liverpool, L69 7ZL, United Kingdom

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ABSTRACT

In this paper we consider a Markov-modulated risk model, where the premium rates, claim frequency and the distribution of the claim sizes vary depending on the state of an external Markov chain. The free reserves of the insurer are invested in a risky asset whose prices are modelled by a geometric Brownian motion, with parameters that are also influenced according to the external Markov process. A system of integro-differential equations for the ruin probabilities and for the expected discounted penalty function is derived. Using Laplace transforms and regular variation theory, we investigate the asymptotic behaviour of both quantities for the case of light or heavy tailed claim size distributions. Specifically, within this setup (where we lose the strong Markov property of the risk process), we show that the ruin probabilities decrease asymptotically as a power function in the case of the light tailed claims, whilst for the heavy tails we show that the probabilities of ruin decay either like a power function, depending on the parameters of the investment, or behave asymptotically like the tails of the claim size distributions.

1. Introduction

The investigation of insurance risk models with stochastic return on investments has attracted a lot of attention in recent years. Stimulated by the paper of Paulsen [1] and Paulsen and Gjessing [2], where continuous time risk processes in a stochastic economic environment are introduced, many researchers have studied Poisson and renewal risk models with risky investments. Lower and upper bounds, numerical solutions, asymptotics and analytic expressions for the probability of ruin (for some individual classes of the aforementioned models), in the case where the wealth process of an insurance portfolio is invested in a stock (whose prices follow a geometric Brownian motion or are Lévy processes), have been derived by several authors. See for example, among others, Cai [3], Cai and Xu [4], Paulsen [5,6], Tang and Tsitsiashvili [7], Tang and Tsitsiashvili [8] and the references therein. More recently, another extension of the aforementioned problem, where a general two sided jump–diffusion risk model that allows correlation between the two Brownian motions driving the insurance risk and investment return, has been investigated by Yin and Wen [9] in the presence of a constant dividend and a threshold barrier strategy.

With regard to the asymptotic results of risk models with investments, Paulsen [10] considers a Lévy risk process compounded by another independent Lévy process and shows asymptotically that, as initial capital increases the ruin probability essentially behaves as a power function of the initial capital. Moreover, Gaier and Grandits [11] showed, within the context of the classical risk model, that when the claim sizes are regularly varying, then the probability of ruin is also regularly varying, whilst Wei [12] extended these results into the context of the renewal risk model. More recently, Hult and Lindskog [13]

* Corresponding author.
E-mail address: L.M.Ramsden@liverpool.ac.uk (L. Ramsden).

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studied the asymptotic decay of finite time ruin probabilities for an insurance portfolio in the presence of heavy-tailed claims when the prices of the risky investments are given by a quite general semimartingale. In this setting, the ruin problem corresponds to determining hitting probabilities for the solution to a randomly perturbed stochastic integral equation. Additionally, Albrecher et al. [14] considered a general class of renewal risk models (where the inter-arrival claim times satisfy an ordinary differential equation with constant coefficients) with geometrical Brownian motion investments and, using regular variation theory, they derived a unified analytic method for the asymptotic behaviour of the probability of ruin. For this general class of renewal risk models with investment, explicit results for the asymptotic ruin probability are given in the case of both light and heavy tailed claims.

The common idea that investing in an asset with stochastic returns proves too risky for an insurance portfolio in the classical risk model, the renewal and the Lévy risk models, can be justified mathematically by all the above papers. However, once we move to non-renewal models (in the sense that the surplus process does not renew itself at the claim time epochs), the strong Markov property is lost and the problem becomes cumbersome. The Markov-modulated risk model was first introduced by Janssen [15] and Reinhard [16] and has since received much attention in the risk theory literature, including applications in queueing theory, see among others Asmussen, Asmussen et al., and Asmussen and O'cinneide [17–19]. The primary motive of these papers is to enhance the flexibility of the models parameter setting. This is achieved by considering an external Markovian environment process which influences both the claim frequencies and the claim severities. The examples usually given are weather conditions, where the sojourns of the external Markov process could be weather types, or in health insurance where the sojourns of the environment process could be certain types of epidemics (see [20]). Surprisingly, only a few authors have studied non-Poissonian risk models in the presence of an investment strategy. Kötter and Bäuerle [21] were the first to introduce a Markov-modulated risk process where risk reserves, under a special investment strategy, can be invested into a stock index following a geometric Brownian motion. Within this setup, for a special class of investment policies, they derive results for the adjustment coefficient. A second study within the Markov-modulated framework was made by Diko and Usábel [22], where they considered a risk model perturbed by diffusion in which the reserves are invested into an asset whose return rate and volatility are time-dependent Markov-modulated. For this model they used Chebyshev's polynomial approximation and Laplace–Carson transforms to obtain a numerical solution for the integro-differential equation system for the risk quantity of interest.

In this paper, we consider a Markov-modulated risk model in which the reserves of the insurance portfolio are continuously invested into an asset whose prices follow a geometrical Brownian motion, which is also influenced by the external Markov chain. For the aforementioned model the Markov property no longer holds and thus the ruin probability is given in terms of an integro-differential equation system. Stimulated by Albrecher et al. [14], we extend their methodology (using Frobenius method for systems—see [23]) to obtain, using regular variation theory, an explicit asymptotic expression of the ultimate ruin probability and the expected discounted penalty function. Within this non-Poissonian model we are able to show that the ruin probability decreases asymptotically as a power function in the case of the light tailed claims, whilst for the heavy tails we show that the probability of ruin decays either like a power function, depending on the parameters of the investment, or behaves asymptotically like the tails of the claim size distributions. The same kind of results hold for the Gerber–Shiu function. Note that the above matrix based analysis holds for more general non-renewal risk models, such as the Markov Arrival Process (MAP) risk models.

In more detail the paper is organised as follows; in Section 2 we introduce a Markov-modulated risk model where the reserves of the insurance portfolio are invested in a risky asset whose price follows a geometrical Brownian motion, in which the drift and volatility parameters are also influenced by the external Markov chain. In Section 3, using the infinitesimal generator argument, we derive an integro-differential equation system for the decompositions of the ruin probabilities. In Section 4, we use Laplace transforms to derive an individual form for the system of ruin probabilities, that will allow an asymptotic analysis in the later sections. In Section 5, we give the general solution for the Laplace system and by using the Frobenius method for matrices, Tauberian theorems and Heaviside Principle, we derive explicit asymptotic expressions for the probabilities of ruin. Section 6 discusses an extension of the methodology used for the ruin probabilities to more general ruin-related quantities, namely the Gerber–Shiu function.

2. Markov-modulated risk process with stochastic investment

In this section, we introduce the Markov-modulated Poisson risk model in the presence of risky asset investment, where the premium rate, the claim arrival rate, the distribution of the claim sizes and the parameters of the return on the surplus investment are influenced by an external Markov chain (see also [21,22]).

Consider the external environment process $\{J(t)\}_{t\geqslant 0}$, which can be interpreted as the general economic conditions that govern the state of the economy. Suppose $\{J(t)\}_{t\geqslant 0}$ is a homogeneous, irreducible and recurrent continuous time Markov process, with finite state space $E = \{1, 2, \ldots, m\}$. Let $\mathbf{Q} = (q_{ij})_{i,j=1}^{m}$, with $q_{ii} = -\sum_{j\neq i}^{m} q_{ij} = -q_i$, for $i \in E$, denote the intensity rate matrix of $\{J(t)\}_{t\geqslant 0}$, with a stationary distribution (which exists and is unique since $\{J(t)\}_{t\geqslant 0}$ is irreducible and has finite state space) given by

$$\pi = (\pi_1, \ldots, \pi_m), \quad \pi_i \geqslant 0,\ i \in E \quad \text{and} \quad \sum_{i\in E} \pi_i = 1.$$