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Extended shift-splitting preconditioners for saddle point problems[☆]

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ABSTRACT

In this paper we consider to solve the linear systems of the saddle point problems by preconditioned Krylov subspace methods. The preconditioners are based on a special splitting of the saddle point matrix. The convergence theory of this class of the extended shift-splitting preconditioned iteration methods is established. The spectral properties of the preconditioned matrices are analyzed. Numerical implementations show that the resulting preconditioners lead to fast convergence when they are used to precondition Krylov subspace iteration methods such as GMRES.

1. Introduction

Suppose that $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ is a matrix of full column rank, and $m \geq n$. Denote by $B^\top$ the transpose of the matrix B. Then the nonsingular saddle point problem is of the form

$$\begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{1.1}$$

where $p \in R^m$ and $q \in R^n$ are two given vectors. Under the above assumptions, the existence and uniqueness of the solution of linear equations (1.1) are guaranteed.

Many practical problems arising from scientific computation and engineering applications require to solve saddle point problem (1.1). For example, computational fluid dynamics [1–4], optimal control [5], the finite element approximation for solving the Navier–Stokes equation [6], constrained optimization [7], the Lagrange-type methods for constrained nonconvex optimization problems [8], weighted least-squares problems [9–11], electronic networks [12], mixed finite element of elliptic PDEs, element-free Galerkin method and so forth; see [13–18] and the references therein. For these practical problems, in general, both A and B in (1.1) are large sparse matrices. Iterative methods become more attractive than direct methods for solving the nonsingular saddle point problem (1.1), although direct methods play an important role in the form of preconditioners embedded in an iterative framework, see [11,19]. Numerical iteration methods for the saddle point problem (1.1) have been studied in many papers, including Uzawa-type methods [3,10,11,19–23], SOR-like methods

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[24–31], RPCG iteration methods [32,33], iterative null space methods [34,35], HSS-type methods [36–40], block triangular and skew-Hermitian splitting methods for positive-definite linear systems [41]. The linear system (1.1) also can be solved using Krylov subspace methods [2,15]. The Krylov subspace methods are more efficient than the stationary iterative methods in general [16]. However, Krylov subspace methods tend to converge slowly when applied to the saddle point problem (1.1), and good preconditioners are key ingredients for the success of Krylov subspace methods in the application. Fortunately, a variety of preconditioners have been proposed and studied in many papers; see [42–44] and their references therein.

In this paper, we introduce a so-called extended shift-splitting (ESS) for the coefficient matrix of saddle point problem (1.1), which produces a class of preconditioners. Based on the ESS splitting, an unconditional convergent fixed-point iteration is proposed, and we call this fixed-point iteration ESS iteration method. Also, the ESS preconditioners we obtained are used in the Krylov subspace methods. For the obtained ESS method, the characteristics of the eigenvalues and eigenvectors of the iteration matrix of this new method are analyzed. Moreover, we also analyze the spectral property of the corresponding preconditioned matrix.

The rest of this paper is organized as follows. In Section 2, the ESS preconditioners for the saddle point problems are introduced and some special ESS preconditioners are analyzed. The convergence properties of the ESS iteration method are studied in Section 3. Moreover, we also study the spectral property of the corresponding preconditioned matrix in the last of this section. In Section 4, some numerical experiments are given to show the feasibility and effectiveness of the ESS preconditioners for the saddle point problems. Finally, we give some brief concluding remarks in Section 5.

Here and in the sequel, for a matrix $C^{m\times n}$, we denote the transpose and the rank of C by $C^\top$ and rank(C), respectively. I is the identity matrix with proper dimension. The 0-matrix is denoted by O. Moreover, the spectral radius of C is denoted by $\rho(C)$. $\|\cdot\|_2$ denotes the l_2 norm of the corresponding vector.

2. The ESS preconditioners

In this section, we present the ESS preconditioners for solving the saddle point problem (1.1), and introduce some special ESS preconditioners.

Denote

$$\mathcal{K}=\begin{pmatrix} A & B \\ -B^\top & O\end{pmatrix},\qquad z=\begin{pmatrix} x\\ y\end{pmatrix},\qquad g=\begin{pmatrix} p\\ -q\end{pmatrix},$$

then (1.1) can be rewritten as

$$\mathcal{K}z=g. \tag{2.1}$$

For the coefficient matrix $\mathcal{K}$ of Eq. (2.1), we make the following ESS splitting (**ESS**):

$$\mathcal{K}=\frac{1}{2}\begin{pmatrix} Q_1+A & B \\ -B^\top & Q_2\end{pmatrix}-\frac{1}{2}\begin{pmatrix} Q_1-A & -B \\ B^\top & Q_2\end{pmatrix}=\mathcal{M}-\mathcal{N}, \tag{2.2}$$

where Q_1 and Q_2 are two symmetric positive definite matrices.

Obviously, because Q_1 and Q_2 are two symmetric positive definite matrices, we can see that $\mathcal{M}$ is a nonsingular matrix. By the special splitting (2.2) for the coefficient matrix $\mathcal{K}$ of Eq. (2.1), the following ESS iteration method can be defined for solving the saddle point problem (1.1):

The ESS method for saddle point problems. Let Q_1 and Q_2 be two symmetric positive definite matrices. Given initial vectors $x^{(0)}\in R^m$, $y^{(0)}\in R^n$. For $k=0,1,2,\ldots$ until the iteration sequence $\{({x^{(k)}}^\top,{y^{(k)}}^\top)^\top\}$ converges to the exact solution of the saddle point problem (1.1), compute

$$\frac{1}{2}\begin{pmatrix} Q_1+A & B \\ -B^\top & Q_2\end{pmatrix}\begin{pmatrix} x^{(k+1)}\\ y^{(k+1)}\end{pmatrix}=\frac{1}{2}\begin{pmatrix} Q_1-A & -B \\ B^\top & Q_2\end{pmatrix}\begin{pmatrix} x^{(k)}\\ y^{(k)}\end{pmatrix}+\begin{pmatrix} p\\ -q\end{pmatrix}. \tag{2.3}$$

It is easy to see that the iteration matrix of the above ESS method is

$$\mathcal{T}=\mathcal{M}^{-1}\mathcal{N}=\begin{pmatrix} Q_1+A & B \\ -B^\top & Q_2\end{pmatrix}^{-1}\begin{pmatrix} Q_1-A & -B \\ B^\top & Q_2\end{pmatrix}. \tag{2.4}$$

The splitting preconditioner that corresponds to the ESS iteration (2.3) is given by

$$\mathcal{P}_{ESS}=\frac{1}{2}\begin{pmatrix} Q_1+A & B \\ -B^\top & Q_2\end{pmatrix},$$

which is called the **ESS preconditioner** for the saddle point matrix $\mathcal{K}$.

With different choices of matrices Q_1 and Q_2, we can easily get a series of splitting preconditioners for the saddle point problem (1.1).