Journal of Computational and Applied Mathematics 313 (2017) 102–118

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Analysis and approximation of a nonlocal obstacle problem[☆]

Qingguang Guan [*], Max Gunzburger

Department of Scientific Computing, Florida State University, Tallahassee FL 32306, United States

ARTICLE INFO

Article history:
Received 19 August 2015
Received in revised form 31 August 2016

Keywords:
Obstacle problems
Nonlocal variational inequality
Fractional Laplacian
Discontinuous
Well posedness
Finite element methods

ABSTRACT

An obstacle problem for a nonlocal operator is considered; the operator is a nonlocal integral analogue of the Laplacian operator and, as a special case, reduces to the fractional Laplacian. In the analysis of classical obstacle problems for the Laplacian, the obstacle is taken to be a smooth function. For the nonlocal obstacle problem considered here, obstacles are allowed to have jump discontinuities. We cast the nonlocal obstacle problem as a minimization problem wherein the solution is constrained to lie above the obstacle. We prove the existence and uniqueness of a solution in an appropriate function space. Then, the well posedness and convergence of finite element approximations are demonstrated. The results of numerical experiments are provided that illustrate the theoretical results and the differences between solutions of local, i.e., partial differential equation, and nonlocal obstacle problems.

1. Introduction

A class of obstacle problems can be cast in the form of an elliptic variational inequality as follows. Given an open domain $\Omega \in \mathbb{R}^n$ with boundary $\partial\Omega$, $f \in L^2(\Omega)$, and $\psi \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\psi \le 0$ on $\partial\Omega$, find u belonging to the closed convex set $\mathcal{K} := \left\{ v \in H_0^1(\Omega) : v \ge \psi \text{ a.e.in } \Omega \right\}$ such that

$$a(u, v-u) \ge (f, v-u) \quad \text{for all } v \in \mathcal{K}, \tag{1}$$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ and $(f,v) = \int_\Omega f v \, dx$. The variational inequality (1) is equivalent to the minimization problem

$$\min_{u\in\mathcal{K}} \Big(\frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega uf \, dx \Big). \tag{2}$$

Obstacle problems of the type (1) or (2) have many applications such as membrane deformation in elasticity theory and nonparametric minimal and capillary surfaces as geometrical problems [1–4]. In general, ψ is assumed to be a smooth (at least continuous) function, in which case it is known that the solution of (1) exists, is unique, is continuous, and possesses Lipschitz continuous first derivatives [1,2,5]. To our knowledge, there are few results about the well-posedness and regularity for obstacle problem with discontinuous obstacle function. Numerical methods for determining approximations of the solution of (1) have also been developed; see, e.g., [1,2,6–8].

Nonlocal obstacle problems arise, e.g., in settings modeled by fractional partial differential equations such as those involving the fractional Laplacian operator [9–13]. In this paper, we treat more general nonlocal problems, with fractional

☆ Supported by the US National Science Foundation under grant number DMS-1013845.
* Corresponding author.
E-mail addresses: qg11b@my.fsu.edu (Q. Guan), mgunzburger@fsu.edu (M. Gunzburger).

http://dx.doi.org/10.1016/j.cam.2016.09.012
0377-0427/

Laplacian and other fractional derivative problems being special cases. Nonlocal operators in the peridynamics theory of solid mechanics [14–16] and anomalous diffusion problems [17–20] also fall within the purview of our study. For nonlocal obstacle problems, one can choose ψ to be a less regular, even discontinuous, function. In addition, because the local, partial differential equation problems are, in a precise sense, the limits of the nonlocal problems we study [19,18,21], one could glean some information about the former for discontinuous ψ by studying the latter.

We define the action of the nonlocal operator $\mathcal{L}$ on a function $u(x) : \Omega \to \mathbb{R}$ by

$$\mathcal{L}u(x) := 2\int_{\mathbb{R}^n} \gamma(x,y)\big(u(x)-u(y)\big)\,dy \quad \forall\, x \in \Omega \subseteq \mathbb{R}^n. \tag{3}$$

The operator $\mathcal{L}$ is deemed *nonlocal* because the value of $\mathcal{L}u$ at a point x requires information about u at points y separated from x by a finite distance; this should be contrasted with, e.g., the *local* Laplacian operator for which the value of Δu at a point x requires information about u only in an infinitesimal neighborhood of x.

The operator $\mathcal{L}$ has a special case the fractional Laplacian operator which is the pseudo-differential operator with Fourier symbol $\mathcal{F}$ given by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\widehat{u}(\xi), \quad 0 < s < 1,$$

where $\widehat{u}$ denotes the Fourier transform of u. Suppose that $u \in L^2(\mathbb{R}^n)$ and that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{n+2s}}dydx < \infty, \quad 0 < s < 1.$$

The vector space of such functions defines the fractional Sobolev space $H^s(\mathbb{R}^n)$. The Fourier transform can be used to show that an equivalent characterization of the fractional Laplacian is

$$(-\Delta)^s u = c_{n,s}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy, \quad 0 < s < 1,$$

for some normalizing constant $c_{n,s}$, see [22,9,23,10,11]. When $\Omega = \mathbb{R}^n$, the fractional Laplacian is the special case of the operator $\mathcal{L}$ defined above for the choice of $\gamma(x,y)$ proportional to $1/|x-y|^{n+2s}$.

When Ω has finite volume we have that the volume constrained minimization problem

$$\min_u\Big(\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dxdy - \int_{\mathbb{R}^n} ufdxdy\Big) \quad \text{subject to } u = 0 \text{ on } \mathbb{R}^n/\Omega \tag{4}$$

is well posed for $0 \le s < 1$, see [18,17]. Note that the volume constraint $u = 0$ on $\mathbb{R}^n/\Omega$ appearing in (4) is needed for well posedness. In fact, the boundary value problem

$$(-\Delta)^s u = g \quad \text{on } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega \tag{5}$$

is not well posed. For $1/2 < s < 1$, solutions are not uniquely defined and for $s \le 1/2$, existence is not, in general, guaranteed. To formulate a well-posed problem, the boundary condition in (5) must be replaced by the volume constraint $u = 0$ on $\mathbb{R}^n/\Omega$; see, e.g., [18]. To differentiate between the two types of constraints, we naturally refer to the constraint $u = 0$ on $\partial\Omega$ in (5) as a boundary condition and refer to the constraint $u = 0$ on $\mathbb{R}^n/\Omega$ in (4) as a *volume constraint*. We also use the latter terminology to refer to constraints of the type $u = 0$ on $\Omega_B \subset \mathbb{R}^n/\Omega$ whenever Ω_B has nonzero volume in $\mathbb{R}^n$.

Here, we treat two types of kernels $\gamma(x,y)$ in (3).

Case 1. For $s \in (0,1)$, $\delta > 0$, $c_{n,s} > 0$, and $x, y \in \mathbb{R}^n$,

$$\gamma_s(x,y) = \begin{cases} \dfrac{c_{n,s}}{|y-x|^{n+2s}} & \text{if } |y-x| \le \delta \\ 0 & \text{if } |y-x| > \delta. \end{cases}$$

Note that $\gamma_s(x,y)$ is non-integrable in Riemann sense.

Case 2. For $l \in (0,n)$, $\delta > 0$, $c_{n,l} > 0$, and $x, y \in \mathbb{R}^n$

$$\gamma_l(x,y) = \begin{cases} \dfrac{c_{n,l}}{|y-x|^{n-l}} & \text{if } |y-x| \le \delta \\ 0 & \text{if } |y-x| > \delta. \end{cases}$$

Note that $\gamma_l(x,y)$ is integrable. Also, note that for both cases the parameter δ, sometimes referred to as the *horizon*, limits the extent of the nonlocal interactions at a point x to the ball of radius δ centered at x.

Let $\Omega_I \subset \mathbb{R}^n$ denote a bounded open domain with piecewise smooth boundary that satisfies the interior cone condition. The corresponding "boundary" domain is defined by

$$\Omega_B := \{y \in \mathbb{R}^n \setminus \Omega_I \text{ such that } \exists\, x \in \Omega_I \text{ such that } \gamma(x,y) \neq 0\},$$