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Numeric solution of Volterra integral equations of the first kind with discontinuous kernels

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ABSTRACT

Numeric methods for solution of the weakly regular linear and nonlinear evolutionary (Volterra) integral equations of the first kind are proposed. The kernels of such equations have jump discontinuities along the continuous curves (endogenous delays) which start at the origin. In order to linearize these equations the modified Newton–Kantorovich iterative process is employed. Two direct quadrature methods based on the piecewise constant and piecewise linear approximation of the exact solution are proposed for linear solutions. The accuracy of proposed numerical methods is $\mathcal{O}(1/N)$ and $\mathcal{O}(1/N^2)$ respectively. A certain iterative numerical scheme enjoying the regularization properties is suggested. Furthermore, generalized numerical method for nonlinear equations is adduced. The midpoint quadrature rule in all the cases is employed. In conclusion several numerical examples are applied in order to demonstrate the efficiency of proposed numerical methods.

Introduction

This article continues the study of novel class of linear Volterra integral equations (VIE) of the first kind with piecewise continuous kernels initiated in [1,2] and pursued in articles [3–6] followed by monograph [7]. The solution of linear integral equations of the first kind is of course classical problem and has been addressed by numerous authors. But only few authors studied these equations in case of jump discontinuous kernels. In general, VIE of the first kind can be solved by reduction to equations of the second kind, regularization algorithms developed for Fredholm equations can be also applied as well as direct discretization methods.

On the other hand, it is known that solutions of integral equations of the first kind can be unstable and this is a well known ill-posed problem. This is due to the fact that the Volterra operator maps the considered solution space into its narrow part only. Therefore, the inverse operator is not bounded. It is necessary to assess the proximity of the solutions and the proximity of the right-hand side using the different metrics. In addition, the proximity of the right-hand side should be in a stronger metric. Moreover, as shown in [7], solutions of the VIE can contain arbitrary constants and can be unlimited as $t \to 0$. Here readers may also refer to [1,3,5,2], where the problems of existence, uniqueness and asymptotic behavior of solutions of equations of this type are explored.

Evolutionary integral equations are in the core of many mathematical models in physics, energetics, economics and ecology. Excellent historical overview of the results concerning the VIEs of the first kind is given by H. Brunner in the paper

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"1896–1996: One hundred years of Volterra integral equations of the first kind" [8]. The theory of integral models of evolving systems was initiated in the works of L. Kantorovich, R. Solow and V. Glushkov in the mid-20th Century. Here readers may refer to the papers of [9,10]. It is well known that Solow publication [10] on vintage capital model led him to the Nobel prize in 1987 for his analysis of economic growth. Such theory employs the VIEs of the first kind where bounds of the integration interval can be functions of time. Here readers may refer e.g. to the monograph [11]. These models take into account the memory of a dynamical system when its past impacts its future evolution. The memory is implemented in the existing technological and financial structure of physical capital (equipment). The memory duration is determined by the age of the oldest capital unit (e.g. equipment) still employed.

The paper [12] is devoted to the construction of iterative numerical algorithm for the systems of nonlinear Volterra-type equations related to the Vintage Capital Models (VCMs) [11]:

$$\begin{cases} x(t) = \displaystyle\int_{y(t)}^{t} H(t,\tau,x(\tau))d\tau, \\ \displaystyle\int_{y(t)}^{t} K(t,\tau,x(\tau))d\tau = f(t), \end{cases} \quad t \in [t_0, T),\ t_0 < T \leqslant \infty,$$

with unknown functions $x(t)$ and $y(t)$ satisfying the initial conditions: $y(t_0) = Y_0 < t_0,\ x(\tau) \equiv \varphi_0(\tau),\ \tau \in (-\infty, t_0]$.

Numerical methods which are optimal with respect to complexity order were constructed in paper [13] for VIEs with certain weakly singular kernels.

First results in studies of the Volterra equations with discontinuous kernels were formulated by G.C. Evans [14] in the beginning of XX century. Results in the spectral theory of integral operators with discontinuous kernels were obtained by A.P. Khromov in his paper [15]. Some results concerning the general approximation theory for integral equations with discontinuous kernels are presented in paper [16].

There are several approaches available for numeric solution of Volterra integral equations of the first kind. One of them is to apply classical regularizing algorithms developed for Fredholm integral equations of the first kind. However, the problem reduces to solving algebraic systems of equations with a full matrix, an important advantage of the Volterra equation is lost and there is a significant increase in arithmetic complexity of the algorithms. The second approach is based on a direct discretization of the initial equations. Here one may face an instability of the approximate solution because of errors in the initial data. The regularization properties of the direct discretization methods are optimal in this sense, where the discretization step is the regularization parameter associated with the error of the source data. However, only low-order quadrature formulas (midpoint quadrature or trapezoidal formulas) are suitable for approximation of the integrals. The Newton–Cotes formulas, Gregory and others (the second order and higher orders) generate divergent algorithms. The detailed description of regularizing direct numerical algorithms is described in the monograph [17]. Here readers may also refer to [18].

It should be noted that it is very difficult to apply these algorithms to solve Eq. (1) in the form of (3) because of the kernel discontinuities (2) as described in Section 1. The adaptive mesh should depend on the curves of the jump discontinuity for each number N of divisions of the considered interval and therefore this mesh cannot be linked to the errors in the source data. It is needed to correctly approximate the integrals.

Below two approaches are proposed for numeric solution for Volterra integral equations of the first kind with piecewise continuous kernels. The first approach is a direct discretization based on piecewise constant and piecewise linear approximations of the exact solution (the first and the second order of accuracy, respectively). The second approach is based on the preliminary determination of the two acceleration values of the unknown function followed by the special regularizing iterative procedure.

The paper is organized into five sections. In Section 1 the problem statement is formulated. The statements concerning the existence and uniqueness of solutions of VIEs with discontinuous kernels are included. Section 2 is dedicated to direct discretization numerical methods based on the piecewise constant and piecewise linear approximation of the exact solution. Section 3 describes the regularization method for linear first kind VIEs of this class. The modified Newton–Kantorovich iterative process for nonlinear VIEs is suggested in Section 4. The numerical examples are given in Section 5.

1. Problem statement

The object of interest is the following integral equation of the first kind

$$\int_0^t K(t,s)x(s)\,ds = f(t), \quad t \in [0,T], \tag{1}$$

where the kernel $K(t,s)$ is discontinuous along continuous curves $\alpha_i(t),\ i = 1, 2, \dots, n-1$, and is of the form

$$K(t,s) = \begin{cases} K_1(t,s), & \alpha_0(t) < s < \alpha_1(t); \\ K_2(t,s), & \alpha_1(t) < s < \alpha_2(t); \\ \dots \\ K_n(t,s), & \alpha_{n-1}(t) < s < \alpha_n(t). \end{cases} \tag{2}$$