Journal of Computational and Applied Mathematics 313 (2017) 129–141

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Existence of solutions for a class of system of functional integral equation via measure of noncompactness

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ARTICLE INFO

*Article history:*
Received 14 February 2016
Received in revised form 10 July 2016

*MSC:*
47H09
47H10
34A12

*Keywords:*
Measure of noncompactness
System of integral equations
Coupled fixed point

ABSTRACT

In this paper, we give some generalization of Darbo's fixed point theorem related to the measure of noncompactness, and present some results on the existence of coupled fixed point theorem for a special class of operators in a Banach space. Our results generalize and extend some comparable results in the literature. Also as an application, we study the existence of solution for a class of the system of nonlinear functional integral equations. Finally an example illustrating the mentioned applicability is also included.



## 1. Introduction

The study of nonlinear integral equations, is a subject of interest for researchers in nonlinear functional analysis. Integral equations occur in many applications, such as in applied mathematics, and also a lot of problems in physics. On the other hand measure of noncompactness is one of the most useful tools in nonlinear and functional analysis, metric fixed point theory and the theory of operator equations in Banach spaces which was first introduced by Kuratowski in [1]. This concept also used to investigate of functional equation, ordinary and partial differential equations, integral and integro-differential equations. In this context several authors have presented some papers on the existence of solution for nonlinear integral equations which involves the use of measure of noncompactness and many other techniques, for instance see [2–21] and [22–28].

In this paper, we apply, the method related to the technique of measures of noncompactness in order to extend the Darbo's fixed point theorem [18] and to generalize some recent results in the literature. In this regard, we state and prove some existence theorems of coupled fixed point for a class of operators in Banach spaces. Moreover, as an application of this theorems, we study the problem of existence of solutions for a class of system of nonlinear integral equations which satisfy in new certain conditions.

## 2. Preliminaries

In this section, we recall some definitions, notations and preliminary results which we will use throughout the paper. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|.\|)$ be a real Banach space with zero element $0$ and $\bar{B}(x, r)$ denote the closed ball in $E$ centered at $x$ with radius $r$. The symbol $\bar{B}_r$ stand for the ball $\bar{B}(0, r)$. If $X$ is a nonempty

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http://dx.doi.org/10.1016/j.cam.2016.09.011

subset of $E$ we denote by $\bar{X}$, $ConvX$ the closure and the closed convex hull of $X$ respectively. Finally, let us denote by $\mathcal{M}_E$ the family of nonempty bounded subsets of $E$ and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

In this paper, we will use axiomatically defined measures of noncompactness as presented in the book [18].

**Definition 2.1** ([18]). A mapping $\mu : \mathcal{M}_E \longrightarrow [0, \infty)$ is said to be a measure of noncompactness in $E$ if it satisfies the following conditions;

(MNC1) The family $Ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $Ker\mu \subseteq \mathcal{N}_E$.
(MNC2) If $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$.
(MNC3) $\mu(\overline{X}) = \mu(X)$.
(MNC4) $\mu(ConvX) = \mu(X)$.
(MNC5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1)$.
(MNC6) If $(X_n)$ is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{n+1} \subseteq X_n$, $(n \geq 1)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $Ker\mu$ described in (MNC1) said to be the kernel of the measure of noncompactness $\mu$. Observe that the intersection set $X_\infty$ from (MNC6) is a member of the family $Ker\mu$. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for any $n$, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in Ker\mu$.

Now we present the definition of a coupled fixed point for a bivariate vector function which we need in the proof of main results and a useful theorem in [18] related to the construction of a measure of noncompactness on a finite product space.

**Definition 2.2** ([29]). An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.

**Theorem 2.3** ([18]). *Suppose $\mu_1, \mu_2, \ldots, \mu_n$ be the measures in Banach spaces $E_1, E_2, \ldots, E_n$ respectively. Moreover assume that the function $F : [0, \infty)^n \to [0, \infty)$ is convex and $F(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots, n$. Then*

$$\widetilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))$$

*defines a measure of noncompactness in $E_1 \times E_2 \times \cdots \times E_n$ where $X_i$ denotes the natural projection of $X$ into $E_i$ for $i = 1, 2, \ldots, n$.*

Now, as a result of Theorem 2.3, we present the following examples.

**Example 2.4.** Let $\mu$ be a measure of noncompactness on a Banach space $E$, and let the function $F : [0, \infty)^2 \to [0, \infty)$ is convex and $F(x_1, x_2) = 0$ if and only if $x_i = 0$ for $i = 1, 2$. Then

$$\widetilde{\mu}(X) = F(\mu(X_1), \mu(X_2))$$

defines a measure of noncompactness in $E \times E$ where $X_i$ denote the natural projection of $X$ into $E$.

**Example 2.5** ([10]). Let $\mu$ be a measure of noncompactness on a Banach space $E$, considering $F(x, y) = x + y$ for any $(x, y) \in [0, \infty)^2$. Then we see that $F$ is convex and $F(x, y) = 0$ if and only if $x = y = 0$, hence all the conditions of Theorem 2.3 are satisfied. Therefore, $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)$ defines a measure of noncompactness in the space $E \times E$ where $X_i$, $i = 1, 2$ denote the natural projections of $X$ into $E$.

**Example 2.6** ([10]). Let $\mu$ be a measure of noncompactness on a Banach space $E$. If we define $F(x, y) = \max\{x, y\}$ for any $(x, y) \in [0, \infty)^2$, then all the conditions of Theorem 2.3 are satisfied, and $\widetilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2)\}$ is a measure of noncompactness in the space $E \times E$ where $X_i$, $i = 1, 2$ denote the natural projections of $X$ into $E$.

Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem, and includes the existence part of Banach's fixed point theorems.

**Theorem 2.7** (Schauder[3]). *Let $\Omega$ be a closed, convex subset of a Banach space $E$. Then every compact, continuous map $T : \Omega \to \Omega$ has at least one fixed point.*

**Theorem 2.8** (Darbo[15]). *Let $\Omega$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$ and let $T : \Omega \longrightarrow \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(T(X)) \leq k\mu(X)$$

*for any $X \subset \Omega$. Then $T$ has a fixed point.*