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# Solving the Yang-Baxter-like matrix equation for rank-two matrices

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ABSTRACT

Let $A = PQ^T$, where $P$ and $Q$ are two $n \times 2$ complex matrices of full column rank such that $\det Q^TP \neq 0$ and so 0 is a semisimple eigenvalue of $A$ with multiplicity $n-2$. We solve the quadratic matrix equation $AXA = XAX$ completely.



## 1. Introduction

In a recent paper [1], the so-called *Yang–Baxter-like matrix equation*

$$AXA = XAX \tag{1.1}$$

was solved completely, where $A = pq^T$ is a given $n \times n$ complex matrix of rank one with two nonzero $n$-dimensional vectors $p$ and $q$. Eq. (1.1) has its origin in the classical Yang–Baxter equation obtained by Yang [2] for studying the many-body problem in 1967 and then by Baxter [3] independently for a lattice model in 1972, and is related to the quantum Yang–Baxter equation. We refer to [4–6] for more details about the Yang–Baxter equation and its relations to other mathematical areas such as braid group and knot theory and physical ones like quantum field theory.

A systematical study of the above quadratic matrix equation from the viewpoint of matrix theory basically began with the paper [7] in which finitely many spectral solutions have been obtained associated with the eigenvalues of an arbitrary matrix $A$, based on the spectral projection matrices onto the generalized eigenspaces of all the eigenvalues of $A$. In addition, based on each spectral solution, infinitely many projection-based solutions were constructed in [8] for the special case that the corresponding eigenvalue is semisimple with multiplicity at least two. For the matrix $A$ with some more general Jordan form structures, several families of solutions have been found in [9], and helped by a well known result on solving the Sylvester equation, the expressions of all or most commuting solutions became available in [10] for the same types of matrices $A$ as in [9]. On the other hand, all the commuting solutions of (1.1) have been determined by [11] when every eigenvalue of $A$ is semisimple so that it is diagonalizable.

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Almost all the solutions obtained so far in the literature are commuting ones, that is, the solutions $X$ of (1.1) satisfy the additional requirement that $AX = XA$, although up to now no formulas of general commuting solutions have been detected for non-diagonalizable matrices $A$. There are much more non-commuting solutions than commuting ones in general. For example, when $A$ is the $2 \times 2$ Jordan block with eigenvalue 1, except for the two obvious solutions $X = 0$ and $X = A$, all the other solutions are non-commuting ones (see the last part of Section 3 of [8]). Therefore, it is necessary to seek general solutions of the Yang–Baxter matrix equation.

Although it looks simple in format, Eq. (1.1) is very difficult to solve for general $A$ since it is equivalent to solving a quadratic system of $n^2$ equations in $n^2$ variables, which is a tough task in algebraic geometry. Consequently, it is still a challenging problem to find all the solutions of (1.1) for general matrices $A$.

In this paper we solve (1.1) for all the solutions when $A$ is of rank two, extending the results of [1] and as an initial step to deal with the general rank-$k$ case. As is well known, any matrix of rank two can be written as

$$A = PQ^T = p_1 q_1^T + p_2 q_2^T,$$

where $P = [p_1, p_2]$ and $Q = [q_1, q_2]$ are $n \times 2$ matrices of full column rank.

Eq. (1.1) always has two solutions $X = 0$ and $X = A$, which are called the trivial solutions. To find all the nontrivial solutions, we assume that the $2 \times 2$ matrix $Q^T P$ is of full rank. As will be seen, there are three cases for our analysis, which will be included in the corresponding subsections in Section 3 that follows.

In the next section we shall present all the solutions of (1.1) when $n = 2$ and $A$ is already its Jordan form with nonzero eigenvalues. Such results will be used in the subsequent section which will be devoted to finding all the solutions with $A = PQ^T$ such that $\det Q^T P \neq 0$. Some numerical examples will be presented in Section 4 to illustrate our results, and we conclude with Section 5.

## 2. Preliminary results

In order to find all the solutions of (1.1), we need to find all the solutions of (1.1) for a $2 \times 2$ matrix $A$ in its Jordan canonical form. Any $2 \times 2$ matrix has one of the following Jordan forms:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad \lambda \neq \mu.$$

Now we list all the solutions of (1.1) when $A$ is one of them and nonsingular.

**Lemma 2.1.** *Let $\lambda \neq 0$ and*

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

*Thenall the nontrivial solutions of (1.1) are*

$$X = \begin{bmatrix} \lambda & y \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} x & \dfrac{\lambda x - x^2}{z} \\ z & \lambda - x \end{bmatrix}, \quad \forall\, x, y, \ \forall\, z \neq 0.$$

**Proof.** Write

$$X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

Then, since $\lambda \neq 0$, the matrix equation (1.1) can be written as

$$\begin{cases} x^2 - \lambda x + yz &= 0, \\ (x + w - \lambda)y &= 0, \\ (x + w - \lambda)z &= 0, \\ w^2 - \lambda w + yz &= 0. \end{cases}$$

First we let $z = 0$. Then the above system is reduced to

$$\begin{cases} x^2 - \lambda x &= 0, \\ (x + w - \lambda)y &= 0, \\ w^2 - \lambda w &= 0, \end{cases}$$

solving which gives the solutions

$$X = 0,\ A,\ \begin{bmatrix} \lambda & y \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y \\ 0 & \lambda \end{bmatrix}, \quad \forall\, y.$$