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A-posteriori error estimation in maximum norm for a strongly coupled system of two singularly perturbed convection–diffusion problems

Li-Bin Liu [a], Yanping Chen [b,*]

[a] College of Mathematics and Statistics, Guangxi Teachers Education University, Nanning, 530001, China

[b] School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China

HIGHLIGHTS

- A maximum norm a posteriori error estimation is presented.
- A monitor function is proposed to design an adaptive grid algorithm.
- We give a sub-algorithm, which is called an arc-length equidistribution algorithm.

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ABSTRACT

A system of two singularly perturbed convection–diffusion problems with strong coupling is studied. This system is discretized by an upwind finite difference scheme on an arbitrary non-uniform mesh. An a posteriori error estimation in the maximum norm is derived. Based on the a posteriori error estimation, a monitor function is proposed to design an adaptive grid algorithm. Especially, to implement the adaptive grid approach, we not only give a mesh generation algorithm, but also give a sub-algorithm, which is called an arc-length equidistribution algorithm. Numerical results are given to confirm our theoretical result.

1. Introduction

Singularly perturbed problems arise in several branches of engineering and applied mathematics. These types of problems are generally used to describe the transport processes involving fluid motion, heat transfer, astrophysics, oceanography, meteorology, semiconductors, hydraulics, pollutant and sediment transport, and chemical reactor [1]. Such problems contain one or more perturbation parameters, which can reflect the physical character of these problems. The solutions of these problems have a multi-scale character. That is, they have two components, one of which changes slowly while the other changes quickly.

For the past decades, many numerical techniques have been proposed for the solution of singularly perturbed convection–diffusion problems; see [2] for a survey. Recently, more attention has been focused on the use of meshes that are adapted to the singularly perturbed problems. One approach is the use of highly non-uniform layer-adapted meshes (see, e.g., [3–5]). Another approach is the use of adaptive meshes generated by equidistributing a monitor function (see, e.g., [6–10]).

* Corresponding author.
E-mail address: yanpingchen@scnu.edu.cn (Y. Chen).

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Nevertheless, for the numerical methods of a coupled system of convection–diffusion problems, there are only a few references: Bellew and O'Riordan [11] considered a coupled system of two singularly perturbed convection–diffusion ordinary differential equations, which is given by

$$\begin{cases} \varepsilon_1 u_1''(x) + a_{1,1}(x)u_1'(x) = f_1(x), \quad x \in (0,1), \\ \varepsilon_2 u_2''(x) + a_{2,2}(x)u_2'(x) + a_{2,1}(x)u_1'(x) = f_2(x), \quad x \in (0,1), \\ a_{1,1}(x) \geq \alpha_1 > 0, \qquad a_{2,2}(x) \geq \alpha_2 > 0, \\ u_1(0), u_1(1), u_2(0), u_2(1) \text{ are given constants,} \end{cases}$$

where functions $a_{i,j}, f_i \in C^3[0,1]$, $i = 1,2, j = 1,2$. Using an upwind finite difference scheme on a piecewise-uniform Shishkin mesh, they established an asymptotic error bound for the numerical approximation $\mathbf{U}$ to the exact solution $\mathbf{u}$ of the form

$$\|\mathbf{u} - \mathbf{U}\|_\infty \leq CN^{-1}(\ln N)^2,$$

where $\|\cdot\|_\infty$ is the standard point-wise maximum norm. In [12], the authors considered a system of $m \geq 2$ convection–diffusion equations in the unknown vector function $\mathbf{u} = (u_1, u_2, \ldots, u_m)^T$. The standard upwind finite difference scheme on an appropriate Shishkin mesh produced a first-order (up to logarithmic factor) parameter-uniform error bound

$$\|\mathbf{u} - \mathbf{U}\|_\infty \leq \begin{cases} CN^{-1}\ln N & \text{if } \sigma_k < 0.5, \\ CN^{-1}(\ln N)^2 & \text{if } \sigma_k = 0.5, \end{cases}$$

where σ_k is a transition point and $\mathbf{U}$ is the numerical solution of $\mathbf{u}$. Cen [13] developed an upwind scheme on a special layer-adapted mesh, so called Shishkin mesh, for a system of two singularly perturbed convection–diffusion problems

$$\begin{cases} -\varepsilon u_1''(x) - a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), \\ -\mu u_2''(x) - a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \\ u_1(0) = u_2(0) = u_1(1) = u_2(1) = 0, \end{cases} \tag{1}$$

where $x \in (0,1)$. He showed that the error in the discrete maximum norm is bounded $CN^{-1}\ln N$. Linß [14] considered the following system of l coupled singularly perturbed convection–diffusion equations:

$$L_k u_k := -\varepsilon_k u_k'' + a_k u_k' + \sum_{m=1}^{l} b_{km} u_m = f_k, \quad x \in (0,1), \qquad u_k(0) = u_k(1) = 0,$$

where $k = 1, 2, \ldots, l$ and a_k, b_{km} and f_k are sufficiently smooth functions. They obtained the error estimates for special layer-adapted meshes as follows:

$$\|\mathbf{u} - \mathbf{U}\|_\infty \leq \begin{cases} CN^{-1}\ln N & \text{for Shishkin meshes,} \\ CN^{-1} & \text{for Bakhvalov meshes.} \end{cases}$$

Recently, O'Riordan and Stynes [15] studied a first-order upwind finite difference scheme for the following problems

$$\begin{cases} \varepsilon u_1'' + a_{11}u_1' + a_{12}u_2' = f_1, \quad x \in (0,1), \\ \varepsilon u_2'' + a_{21}u_1' + a_{22}u_2' = f_2, \quad x \in (0,1), \\ u_j(0) = d_{j,0}, u_j(1) = d_{j,1}, \quad j = 1,2, \end{cases}$$

where $0 < \varepsilon \ll 1$, functions a_{ij} and f_j are assumed to lie in $C^1[0,1]$, and $d_{j,i} (i,j = 0,1)$ are given constants. They established uniform convergence of $O(N^{-1}\ln N)$ on a piecewise-uniform Shishkin mesh.

As far as we know, the layer-adapted grid approaches have all been used and analyzed in the above literatures devoted to a coupled system of singularly perturbed convection–diffusion problem. However, amidst all this activity, few authors have paid attention to the numerical analysis of these problems by using the adaptive grid method. Recently, Liu and Chen [16] also considered the above problem (1) and presented a robust adaptive grid method which was first-order convergent, independent of perturbation parameters. Linß [17] developed an adaptive numerical method to the following problem

$$\mathcal{L}\mathbf{u} := -\text{diag}(\varepsilon)\mathbf{u}'' - B\mathbf{u}' + A\mathbf{u} = \mathbf{f} \quad \text{in } (0,1), \qquad \mathbf{u}(0) = \mathbf{u}(1) = \mathbf{0}, \tag{2}$$

where the small perturbation parameters $\varepsilon_i \in (0,1]$, $i = 1, \ldots, l$, $\mathbf{u} = (u_1, \ldots, u_l)^T \in (C^2(0,1) \cap C[0,l])^l$. Based on the a posteriori error estimation, he obtained a positive monitor function and designed an adaptive grid algorithm. However, we think that the monitor function and the adaptive grid algorithm proposed in [17] are much more complex. Therefore, it is necessary to provide an efficient monitor function and the corresponding adaptive algorithm in a simplex way.

In this paper, we will develop an adaptive grid approach to solve the following strongly coupled system of singularly perturbed convection–diffusion problem of non-conservative form

$$\begin{cases} \mathcal{L}\mathbf{u} := E\mathbf{u}'' + A\mathbf{u}' = \mathbf{0}, \\ \mathbf{u}(0) = \mathbf{d}_0, \qquad \mathbf{u}(1) = \mathbf{d}_1, \end{cases} \tag{3}$$