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Error analysis of a fractional-step method for magnetohydrodynamics equations

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ABSTRACT

This paper focuses on a fractional-step finite element method for the magnetohydrodynamics problems in three-dimensional bounded domains. It is shown that the proposed fractional-step scheme allows for a discrete energy identity. A rigorous error analysis is presented. We derive the temporal and spatial error estimates of $\mathcal{O}(\Delta t + h)$ for the velocity and the magnetic field in the discrete space $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ under the constraint $\Delta t \geq Ch$.

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1. Introduction

The incompressible magnetohydrodynamics (MHD) problems are used to describe the flow of a viscous, incompressible and electrically conducting fluid, which are governed by the following time-dependent nonlinear coupled problems:

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S\mathbf{b}\times \text{curl}\,\mathbf{b} = \mathbf{f}, \tag{1.1}$$

$$\text{div}\,\mathbf{u} = 0, \tag{1.2}$$

$$\frac{\partial \mathbf{b}}{\partial t} + \frac{1}{Rm}\text{curl}\,(\text{curl}\,\mathbf{b}) - \text{curl}\,(\mathbf{u}\times\mathbf{b}) = 0, \tag{1.3}$$

$$\text{div}\,\mathbf{b} = 0, \tag{1.4}$$

for $x \in \Omega$ and $t \in (0, T)$ with $T > 0$, where $\Omega \subset \mathbf{R}^3$ is a bounded and simply-connected domain which is either convex or has a $C^{1,1}$ boundary $\partial\Omega$. Re, Rm and S are three positive constants and denote the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. The vector-value function $\mathbf{f}$ represents the body forces applied to the fluid. The MHD problems (1.1)–(1.4) couple the incompressible Navier–Stokes equations with Maxwell's equations. Thus, the unknowns in (1.1)–(1.4) are the fluid velocity $\mathbf{u}$, the pressure p and the magnetic field $\mathbf{b}$. We refer to Hughes [1] and Moreau [2] for the understanding of the physical background of the MHD problems. To study (1.1)–(1.4), the appropriate initial and boundary conditions are needed. For the sake of simplicity, in this paper, we consider the following initial and

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boundary conditions:

$$\mathbf{u}(x,0)=\mathbf{u}_0,\qquad \mathbf{b}(x,0)=\mathbf{b}_0\quad \text{in } \Omega, \tag{1.5}$$

$$\mathbf{u}=0,\qquad \mathbf{b}\cdot\mathbf{n}=0,\qquad \text{curl}\,\mathbf{b}\times\mathbf{n}=0\quad \text{on } \partial\Omega\times[0,T], \tag{1.6}$$

where $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$. It is necessary to require that $\mathbf{u}_0$ and $\mathbf{b}_0$ satisfy the compatibility conditions div $\mathbf{u}_0=0$ and div $\mathbf{b}_0=0$.

The numerical methods for the incompressible MHD problems have received much attention in the last decades. We refer to Gerbeau–Bris–Lelièvre [3] for a review of numerical methods. The mixed finite element approximation was first proposed and studied for the stationary MHD problems in [4], where $\mathbf{H}^1$-conforming elements were used to discretize the magnetic field provided that Ω is either convex or has a $C^{1,1}$ boundary $\partial\Omega$. Inspired by the stabilization method for Stokes problems in [5], a stabilization mixed finite element method for the stationary MHD problems was developed by Gerbeau [6]. For the time-dependent MHD problems (1.1)–(1.6), He proposed a linearized semi-implicit Euler scheme in [7], where $\mathbf{L}^2$-unconditional convergence was proved by using the negative norm technique. For the non-convex domain or Lipschitz polyhedra domain of engineering practice, the magnetic field $\mathbf{b}$ may have regularity below $\mathbf{H}^1(\Omega)$. In this case, the $\mathbf{H}^1$-conforming finite element discretization for $\mathbf{b}$, albeit stable, may not converge to the correct magnetic field. A mixed finite element formulation based on $\mathbf{H}$(curl)-elements (or Nédélec elements) for $\mathbf{b}$ was proposed and studied by Schötzau in [8] for the stationary MHD problems. Other different numerical methods can be found in [9–18] and references cited therein. Roughly speaking, the difficulties in solving the MHD problems numerically are mainly of three kinds: the mixed type of the equations; the incompressible condition and the treatment of the pressure; the nonlinearities of the problems, which are very similar to the incompressible Navier–Stokes problems. In the 1960s, Chorin [19] and Temam [20] proposed a projection method for Navier–Stokes problems, which decoupled the velocity and the pressure in the Navier–Stokes problems. The idea of the projection method is first to compute a velocity field without taking into account incompressibility, and then perform a pressure correction, which is a projection back to the subspace of solenoidal (divergence-free) vector fields. However, the drawback is the appearance of the numerical boundary layer due to the incompatibility of the pressure boundary conditions with those of the original Navier–Stokes problems [21,22]. For the time-dependent MHD problems (1.1)–(1.6), inspired by the projection method in [19,20], Prohl in [14] proposed a projection scheme. However, the projection scheme in [14] does not allow for a discrete energy estimate. To avoid using artificial boundary conditions of pressure type, some fractional-step schemes for the Navier–Stokes problems were introduced and studied in [23,24]. It is a two-step scheme in which the incompressibility and the nonlinearities of the Navier–Stokes problems are split into different steps, and allows the enforcement of the original boundary conditions in all substeps.

In this paper, we propose a two-step fractional-step scheme to approximate the solution of the MHD problems (1.1)–(1.4) with the initial and boundary conditions (1.5)–(1.6). We will prove that the proposed fractional-step scheme allows for a discrete energy identity. To state the main results derived, we introduce the following notations. Let X be a Banach space equipped with norm $\|\cdot\|_X$. Let $0=t_0<t_1<\cdots<t_N=T$ be a uniform partition of the time interval $[0,T]$ with time step $\Delta t=T/N$ and $t_n=n\Delta t$ for $0\le n\le N$. We denote two discrete norms by

$$\|\mathbf{u}^n\|_{l^2(X)}=\left(\Delta t\sum_{n=1}^{N}\|\mathbf{u}^n\|_X^2\right)^{1/2},\qquad \|\mathbf{u}^n\|_{l^\infty(X)}=\max_{1\le n\le N}\|\mathbf{u}^n\|_X.$$

It is proved that the time-discrete fractional-step scheme provides the temporal error estimates of $\mathcal{O}(\Delta t)$ for the velocity and the magnetic field in $l^2(\mathbf{H}^1)\cap l^\infty(\mathbf{L}^2)$ and $\mathcal{O}(\sqrt{\Delta t})$ for the pressure in $l^2(L^2)$. For the fully discrete fractional-step scheme, the finite element error estimates of $\mathcal{O}(\Delta t+h)$ for the velocity and the magnetic field in $l^2(\mathbf{H}^1)\cap l^\infty(\mathbf{L}^2)$ are obtained under the constraint $\Delta t\ge Ch$.

The remainder of this paper is organized as follows: in the next section, we begin with some notations, lay out some assumptions and recall some known inequalities frequently used. The new linearized projection scheme and the main results are presented in Section 3. Meanwhile, the discrete energy identity is derived in Section 3. The proof containing the main results is given in Sections 4 and 5, which is split into several theorems.

2. Mathematical setting

For the mathematical setting of the MHD problems (1.1)–(1.4) with the initial and boundary conditions (1.5)–(1.6), we introduce some function spaces and their associated norms. For $k\in\mathbb{N}^+$, $1\le s\le+\infty$, let $W^{k,s}(\Omega)$ denote the standard Sobolev space. The norm in $W^{k,s}(\Omega)$ is defined by

$$\|u\|_{W^{k,s}}=\begin{cases}\left(\displaystyle\sum_{|\beta|\le k}\int_\Omega|\nabla^\beta u|^s dx\right)^{1/s} & \text{for } 1\le s<+\infty,\\ \displaystyle\sum_{|\beta|\le k}\sup_\Omega|\nabla^\beta u| & \text{for } s=+\infty,\end{cases}$$

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