Journal of Computational and Applied Mathematics 313 (2017) 185–201

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Trigonometric collocation methods based on Lagrange basis polynomials for multi-frequency oscillatory second-order differential equations

Bin Wang [a,*], Xinyuan Wu [b], Fanwei Meng [a]

[a] School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, PR China
[b] Department of Mathematics, Nanjing University, Nanjing 210093, PR China

ARTICLE INFO

Article history:
Received 8 April 2016
Received in revised form 6 September 2016

MSC:
65L05
65L06
34C15
34E05

Keywords:
Trigonometric collocation methods
Lagrange polynomials
Multi-frequency oscillatory second-order systems
Variation-of-constants formula

ABSTRACT

In the present work, a kind of trigonometric collocation methods based on Lagrange basis polynomials is developed for effectively solving multi-frequency oscillatory second-order differential equations $q''(t)+Mq(t)=f(q(t))$. The properties of the obtained methods are investigated. It is shown that the convergent condition of these methods is independent of $\|M\|$, which is very crucial for solving oscillatory systems. A fourth-order scheme of the methods is presented. Numerical experiments are implemented to show the remarkable efficiency of the methods proposed in this paper.

1. Introduction

The numerical treatment of multi-frequency oscillatory systems is a computational problem of an overarching importance in a wide range of applications, such as quantum physics, circuit simulations, flexible body dynamics and mechanics (see, e.g. [1–7] and the references therein). The main theme of the present paper is to construct and analyse a kind of efficient collocation methods for solving multi-frequency oscillatory second-order differential equations of the form

$$q''(t)+Mq(t)=f\big(q(t)\big),\quad q(0)=q_0,\ q'(0)=q_0',\ t\in[0,t_{\text{end}}],\tag{1}$$

where M is a $d\times d$ positive semi-definite matrix implicitly containing the frequencies of the oscillatory problem and $f:\mathbb{R}^d\to\mathbb{R}^d$ is an analytic function. The solution of this system is a multi-frequency nonlinear oscillator because of the presence of the linear term Mq. System (1) is a highly oscillatory problem when $\|M\|\gg 1$. In recent years, various numerical methods for approximating solutions of oscillatory systems have been developed by many researchers. Readers are referred to [8–16] and the references therein. Once it is further assumed that M is symmetric and f is the negative gradient of a

* Corresponding author.
E-mail addresses: wangbinmaths@gmail.com (B. Wang), xywu@nju.edu.cn (X. Wu), fwmeng@mail.qfnu.edu.cn (F. Meng).

http://dx.doi.org/10.1016/j.cam.2016.09.017
0377-0427/

real-valued function $U(q)$, the system (1) is identical to the following initial value Hamiltonian system

$$\begin{aligned} \dot{q} &= \nabla_p H(q, p), \quad q(0) = q_0, \\ \dot{p} &= -\nabla_q H(q, p), \quad p(0) = p_0 \equiv q_0' \end{aligned} \tag{2}$$

with the Hamiltonian function

$$H(q, p) = \frac{1}{2}p^\intercal p + \frac{1}{2}q^\intercal Mq + U(q). \tag{3}$$

This is an important system which has received much attention by many authors (see, e.g. [1,2,4,5,17]).

In [18], the authors took advantage of shifted Legendre polynomials to obtain a local Fourier expansion of the system (1) and derived a kind of collocation methods (trigonometric collocation methods). The analysis and the results of numerical experiments in [18] showed that the trigonometric collocation methods are more efficient in comparison with some alternative approaches that have previously appeared in the literature. Motivated by the work in [18], this paper is devoted to the formulation and analysis of another trigonometric collocation methods for solving multi-frequency oscillatory second-order systems (1). We will consider a more classical approach and use Lagrange polynomials to obtain the methods. Because of this different approach, compared with the methods in [18], the obtained methods have a simpler scheme and can be implemented in practical computations at a lower cost. These trigonometric collocation methods are designed by interpolating the function f of (1) by Lagrange basis polynomials, and incorporating the variation-of-constants formula with the idea of collocation methods. It is noted that these integrators are a kind of collocation methods and they share all the interesting features of collocation methods. We analyse the properties of the trigonometric collocation methods. We also consider the convergence of the fixed-point iteration for the methods. It is important to emphasize that for the trigonometric collocation methods, the convergent condition is independent of $\|M\|$, which is a very important property for solving oscillatory systems.

This paper is organized as follows. In Section 2, we formulate the scheme of trigonometric collocation methods based on Lagrange basis polynomials. The properties of the obtained methods are analysed in Section 3. In Section 4, a fourth-order scheme of the methods is presented and numerical tests confirm that the method proposed in this paper yields a dramatic improvement. Conclusions are included in Section 5.

2. Formulation of the methods

To begin with we restrict the multi-frequency oscillatory system (1) to the interval $[0, h]$ with any $h > 0$:

$$q''(t) + Mq(t) = f\big(q(t)\big), \quad q(0) = q_0, \; q'(0) = q_0', \; t \in [0, h]. \tag{4}$$

With regard to the variation-of-constants formula for (1) given in [19], we have the following result on the exact solution $q(t)$ of the system (1) and its derivative $q'(t) = p(t)$:

$$\begin{aligned} q(t) &= \phi_0(t^2M)q_0 + t\phi_1(t^2M)p_0 + t^2\int_0^1 (1-z)\phi_1\big((1-z)^2t^2M\big)f\big(q(tz)\big)dz, \\ p(t) &= -tM\phi_1(t^2M)q_0 + \phi_0(t^2M)p_0 + t\int_0^1 \phi_0\big((1-z)^2t^2M\big)f\big(q(tz)\big)dz, \end{aligned} \tag{5}$$

where $t \in [0, h]$ and

$$\phi_i(M) := \sum_{l=0}^{\infty} \frac{(-1)^l M^l}{(2l+i)!}, \quad i = 0, 1. \tag{6}$$

From this result, it follows that

$$\begin{aligned} q(h) &= \phi_0(V)q_0 + h\phi_1(V)p_0 + h^2\int_0^1 (1-z)\phi_1\big((1-z)^2V\big)f\big(q(hz)\big)dz, \\ p(h) &= -hM\phi_1(V)q_0 + \phi_0(V)p_0 + h\int_0^1 \phi_0\big((1-z)^2V\big)f\big(q(hz)\big)dz, \end{aligned} \tag{7}$$

where $V = h^2M$.

The main point in designing practical schemes to solve (1) is based on replacing $f(q)$ in (7) by some expansion. In this paper, we interpolate $f(q)$ as

$$f\big(q(\xi h)\big) \sim \sum_{j=1}^{s} l_j(\xi)f\big(q(c_jh)\big), \quad \xi \in [0, 1], \tag{8}$$