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A weak Local Linearization scheme for stochastic differential equations with multiplicative noise

J.C. Jimenez [a,*], C. Mora [b], M. Selva [c]

[a] Instituto de Cibernética, Matemática y Física, La Habana, Cuba
[b] Departamento de Ingeniería Matemática and CI^{2}MA, Universidad de Concepción, Chile
[c] Departamento de Ingenería Matemática, Universidad de Concepción, Chile

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ABSTRACT

In this paper, a weak Local Linearization scheme for Stochastic Differential Equations (SDEs) with multiplicative noise is introduced. First, for a time discretization, the solution of the SDE is locally approximated by the solution of the piecewise linear SDE that results from the Local Linearization strategy. The weak numerical scheme is then defined as a sequence of random vectors whose first moments coincide with those of the piecewise linear SDE on the time discretization. The scheme is explicit, preserves the first two moments of the solution of SDEs with linear drift and diffusion coefficients in state and time, and inherits the mean-square stability or instability that such solution may have. The rate of convergence is derived and numerical simulations are presented for illustrating the performance of the scheme.

1. Introduction

During 30 years the class of Local Linearization integrators has been developed for different types of deterministic and random differential equations. The essential principle of such integration methods is the piecewise linearization of the given differential equation to obtain consecutive linear equations that are explicitly solved at each time step. This general approach has worked well for the classes of ordinary, delay, random and stochastic differential equations. Key element of such success is the use of explicit solutions or suitable approximations for the resulting linear differential equations. Precisely, the absence of explicit solution or adequate approximation for linear Stochastic Differential Equations (SDEs) with multiplicative noise is the main reason of the limited application of the Local Linearization approach to nonlinear SDEs with multiplicative noise. For these equations, the available Local Linearization integrators are of two types: those introduced in [1] for scalar equations and those considered in [2–4]. The former uses the explicit solution of the scalar linear equations with multiplicative noise, while the latter employs the solution of the linear equation with additive noise that locally approximates the nonlinear equation.

Directly related to the development of the Local Linearization integrators is the concept of Local Linear approximations (see, e.g., [5–7]). These approximations to the solution of the differential equations are defined as the continuous time solution of the piecewise linear equations associated to the Local Linearization method. These continuous approximations have played a fundamental role for studying the convergence, stability and dynamics of the Local Linearization integrators for all the classes of differential equations mentioned above with the exception of the SDEs with multiplicative noise. For

* Corresponding author.
E-mail addresses: jcarlos@icimaf.cu (J.C. Jimenez), cmora@ing-mat.udec.cl (C. Mora), selva@ing-mat.udec.cl (M. Selva).

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this last class of equations, the Local Linear approximations have only been used for constructing piecewise approximations to the mean and variance of the states in the framework of continuous–discrete filtering problems (see [7]).

The purpose of this work is to construct a weak Local Linearization integrator for SDEs with multiplicative noise based on suitable weak approximation to the solution of piecewise linear SDEs with multiplicative noise. For this, we cross two ideas: (1) as in [7], the use of the Local Linear approximations for obtaining piecewise approximations to the mean and variance of the SDEs with multiplicative noise; and (2) as in [8], at each integration step, the generation of a random vector with the mean and variance of the Local Linear approximation at this integration time. For implementing this, new formulas recently obtained in [9] for the mean and variance of the solution of linear SDEs with multiplicative noise are used, which are computationally more efficient than those formerly proposed in [10,7]. Notice that this integration approach is conceptually different to that usually employed for designing weak integrators for SDEs. Typically, these integrators are derived from a truncated Ito–Taylor expansion of the equation's solution at each integration step, and include the generation of random variables with moments equal to those of the involved multiple Ito integrals [11,12].

The paper is organized as follows. After some basic notations in Section 2, the new Local Linearization integrator is introduced in Section 3. Its rate of convergence is derived in Section 4 and, in the last section, numerical simulations are presented in order to illustrate the performance of the numerical integrator.

2. Basic notations

Let us consider the SDE with multiplicative noise

$$X_t = X_{t_0} + \int_{t_0}^{t} f\left(s, X_s\right) ds + \sum_{k=1}^{m} \int_{t_0}^{t} g^k\left(s, X_s\right) dW_s^k, \quad \forall t \in [t_0, T], \tag{1}$$

where $f, g^k : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ are smooth functions for all $k = 1, \ldots, m$, $W^1, \ldots, W^m$ are independent Wiener processes on a filtered complete probability space $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq t_0}, \mathbb{P}\right)$, and X_t is an adapted $\mathbb{R}^d$-valued stochastic process. In addition, let us assume the usual conditions for the existence and uniqueness of a weak solution of (1) with bounded moments (see, e.g., [11]).

Throughout this paper, we consider the time discretization $t_0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ with $\tau_{n+1} - \tau_n \leq \Delta$ for all $n = 0, \ldots, N-1$ and $\Delta > 0$. We use the same symbol $K(\cdot)$ (resp., K) for different positive increasing functions (resp., positive real numbers) having the common property to be independent of $(\tau_k)_{k=0,\ldots,N}$. Moreover, $A^\top$ stands for the transpose of the matrix A, and $|\cdot|$ denotes the Euclidean norm for vectors. $\mathcal{C}_P^\ell\left(\mathbb{R}^d, \mathbb{R}\right)$ denotes the collection of all ℓ-times continuously differentiable functions $g : \mathbb{R}^d \to \mathbb{R}$ such that g and all its partial derivatives of orders $1, 2, \ldots, \ell$ have at most polynomial growth. A random variable η with mean $\overline{\eta}$ will be called symmetric around the mean $\overline{\eta}$ if $\eta - \overline{\eta}$ and $-(\eta - \overline{\eta})$ have the same distribution.

3. Numerical method

Suppose that $z_n \approx X_{\tau_n}$ with $n = 0, \ldots, N-1$. Set $g^0 = f$. Taking the first-order Taylor expansion of g^k yields

$$g^k(t, x) \approx g^k(\tau_n, z_n) + \frac{\partial g^k}{\partial x}(\tau_n, z_n)(x - z_n) + \frac{\partial g^k}{\partial t}(\tau_n, z_n)(t - \tau_n)$$

whenever $x \approx z_n$ and $t \approx \tau_n$ for all $k = 0, \ldots, m$. Therefore

$$X_t \approx z_n + \sum_{k=0}^{m} \int_{\tau_n}^{t} \left(B_n^k X_s + b_n^k(s)\right) dW_s^k \quad \forall t \in [\tau_n, \tau_{n+1}],$$

with $W_s^0 = s$, $B_n^k = \frac{\partial g^k}{\partial x}(\tau_n, z_n)$ and

$$b_n^k(s) = g^k(\tau_n, z_n) - \frac{\partial g^k}{\partial x}(\tau_n, z_n) z_n + \frac{\partial g^k}{\partial t}(\tau_n, z_n)(s - \tau_n). \tag{2}$$

This follows that, for all $t \in [\tau_n, \tau_{n+1}]$, X_t can be approximated by

$$Y_t = z_n + \sum_{k=0}^{m} \int_{\tau_n}^{t} \left(B_n^k Y_s + b_n^k(s)\right) dW_s^k, \quad \forall t \in [\tau_n, \tau_{n+1}], \tag{3}$$

which is the first order Local Linear approximation of X_t used in [7]. From (3) we have that $\mathbb{E}\phi\left(X_{\tau_{n+1}}\right) \approx \mathbb{E}\phi\left(Y_{\tau_{n+1}}\right)$ for any smooth function ϕ, and so $X_{\tau_{n+1}}$ might be weakly approximated by a random variable z_{n+1} such that the first moments of $z_{n+1} - z_n$ be similar to those of $Y_{\tau_{n+1}} - z_n$. This leads us to the following Local Linearization scheme.