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Valuing American floating strike lookback option and Neumann problem for inhomogeneous Black–Scholes equation

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ABSTRACT

This paper presents our study of American floating strike lookback options written on dividend-paying assets. The valuation of these options can be mathematically formulated as a free boundary inhomogeneous Black–Scholes PDE with a Neumann boundary condition, which we, by using a Mellin transform, convert into a relatively simple ordinary differential equation with Dirichlet boundary conditions. We then use these results to derive an integral equation that can be used to calculate the price of American floating strike lookback options. In addition, we also used Mellin transforms to derive the closed-form of the perpetual case.

1. Introduction

Lookback options are path-dependent options with payoffs depending on the maximum or the minimum of the underlying asset price during the lifetime of the option. A popular form of lookback options in the insurance field is equity-indexed annuities (EIA), although other kinds of lookback options are also traded worldwide in the exchange market (refer to [1,2] for further details on the subject and related topics.) Various researchers have published results relating to the pricing of European lookback options; for example, Goldman et al. [3] and Conze and Viswanathan [4] derived the exact formula for the value function of European lookback options and Dai et al. [5] presented a formula for quanto lookback options regarding two underlying assets.

As American option holders can exercise their options at any instant before expiry, the early exercise policy should be considered when valuing American options. This is the reason why problems involving American options are usually referred to as optimal stopping problems or free boundary problems. Regarding American option theories, Kwok [6] gave an elaborate description, whereas Peskir and Shiryaev [7] established a number of theories related to optimal stopping problems. Furthermore, I. Kim [8] derived an integral equation satisfied by American options, because the closed-form solution of the American option did not yet exist at the time.

American lookback options can be thought of as a combination of American options and lookback options. Therefore, they have the properties of both of these types of options. Especially, valuing them requires a solution for the free boundary problems, an approach which is similar to the valuation of other American options. In addition, the presence of a lookback state variable results in a Neumann boundary condition. American lookback options can be divided into two categories: American fixed lookback options and American floating strike lookback options. Both of these types of options solve the same

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partial differential equation (PDE), but their payoff functions are different. There is also a relationship between American fixed strike lookback options and Russian options, where the latter could be considered a kind of perpetual version of the former. Anyone interested in Russian options can refer to [9–11]. The distinctive property of American floating strike lookback options is the homogeneity of their value functions. Such homogeneity makes it possible to reduce the dimension of the problem by one; thus, the structure of American floating strike lookback options is relatively simpler than that of the usual American fixed strike options. We focus on American floating strike options in this paper.

American floating strike lookback options have been studied previously. For example, Yu et al. studied the exercise boundary of American floating strike lookback options [12], and Dai and Kwok characterized the optimal stopping region of American lookback options [13,14]. Lai and Lim [15] proposed a way to calculate the value of American floating strike lookback options by using a numerical approach known as the Bernoulli walk approach. Kimura performed a premium decomposition for American floating lookback options employing Laplace transforms [16]. Finally, we remark that Dai succeeded in obtaining a closed-form solution of American options [17].

In this work, our approach was to mainly use the Mellin transform, which is a type of integral transform that can be considered a two-sided Laplace transform. Especially, a Mellin transform is widely used in solving option problems because it can be used to convert a Black–Scholes PDE into a simple ordinary differential equation (ODE). Remarkable results have been achieved by following the approach based on the Mellin transform; for example, Panini and Srivastav priced European, American, as well as perpetual American options using Mellin transforms [18,19], respectively. Frontczak [20] defined a modified Mellin transform and used it to derive an integral equation satisfied by an American call option. Yoon and Kim obtained a closed form of vulnerable options using double Mellin transforms [21]. Yoon also obtained a solution for European options with a stochastic interest model [22]. Buchen [23] analyzed the pricing of lookback type options and Jeon et al. [24] obtained integral equation representation of Russian option with finite time horizon by using Mellin transform techniques, respectively. In addition, Jeon et al. [25] derived a closed form solution of vulnerable geometric Asian options by utilizing double Mellin transforms.

This paper consists of five parts. In the first part (Section 2), we review the concepts of American floating lookback options and formulate the equivalent PDE problems. In the second part (Section 3), we derive the general solution of the inhomogeneous Black–Scholes equation with given Neumann boundary conditions with the aid of Mellin transform techniques. In the third part (Section 4), we apply the results of Section 3 to American floating lookback options to obtain the integral equation representation for American floating lookback options. In the fourth part (Section 5), we analyze the solution obtained in Section 4 to derive the closed form of the value function of perpetual American options. In the fifth part (Section 6), we summarize the work we have done. We added Appendix in which we summarize the basic definition, properties, and lemma regarding the Mellin transform for those who are not familiar with it.

2. Model formulation

Let S_t denote the underlying asset of the floating strike lookback option under a risk-neutral probability measure $\mathbb{P}$.

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t \quad (r > q)$$

where $r(>0)$ is the riskless interest rate, σ and $q(>0)$ are the volatility and dividend yield of X, respectively, and W_t is a one-dimensional standard Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0} \equiv \mathbb{F}$ is the natural filtration generated by $\mathbb{F}$.

For the process $(S_t)_{t\geq 0}$, define the minimum process as

$$m_t = \min_{0\leq \gamma \leq t} S_\gamma, \quad t \geq 0.$$

Consider an *American floating strike lookback call* option with a given finite time horizon $T > 0$. The payoff at maturity is given by $(S_T - m_T)$. In the absence of arbitrage opportunities, the value $C(t, S_t, m_t)$ is a solution of an *optimal stopping problem* (see [7])

$$C(t,s,m) = \sup_{\tau\in[t,T]} \mathbb{E}\left[e^{-r(\tau-t)}(S_\tau - m_\tau) \mid S_t = s, m_t = m\right] \tag{2.1}$$

where τ is the stopping time of the filtration $\mathbb{F}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$.

It is known that the *optimal stopping problem* (2.1) can be reduced to a *free boundary problem*. Define the differential operator $\mathcal{L}$ by

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + (r-q)s\frac{\partial}{\partial s} - r.$$

Then, the free boundary problem can be written in a linear complementary form (see [6,26]) as

$$\begin{aligned} &\mathcal{L}\, C(t,s,m) \leq 0, \quad C(t,s,m) \geq S - m, \\ &(\mathcal{L}\, C(t,s,m))\,(C(t,s,m) - (s-m)) = 0, \quad s > m > 0,\ 0 \leq t < T, \end{aligned} \tag{2.2}$$