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Certain integrals involving the generalized hypergeometric function and the Laguerre polynomials

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ABSTRACT

The aim of the paper is to establish certain new integrals involving the generalized Gauss hypergeometric function, generalized confluent hypergeometric function, and the Laguerre Polynomials. On account of the most general nature of the functions involved therein, our main findings are capable of yielding a large number of new, interesting, and useful integrals, expansion formulas involving the hypergeometric function, and the Laguerre Polynomials as their special cases.

1. Introduction and definitions

Recently, some fundamental properties and characteristics of the generalized Beta type function

$$B_p^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}\,t \tag{1.1}$$

for

$$\Re(p)\ge 0,\qquad \min\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta)\}>0,\quad \text{and}\quad B_0^{(\alpha,\beta)}(x,y)=B(x,y)$$

were introduced and studied in [1], where

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}\,t$$

for $\Re(x)>0$ and $\Re(y)>0$ is the well-known Euler Beta function. See also [2, p. 32, Chapter 4] and [1, p. 4602,(4)].

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Along with the generalized Beta function (1.1), the potentially useful generalized Gauss hypergeometric functions

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty} a_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{1.2}$$

and the generalized confluent hypergeometric functions

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}$$

for $|z| < 1$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$, $\Re(c) > \Re(b) > 0$, and $\Re(p) \geq 0$ were also introduced and studied. See [2, p. 39, Chapter 4] and [1, p. 4606, Section 3].

When $p = 0$, the functions $F_p^{(\alpha,\beta)}(a,b;c;z)$ and ${}_1F_1^{(\alpha,\beta;p)}(b;c;z)$ would reduce immediately to the extensively-investigated Gauss hypergeometric functions ${}_2F_1(\cdot)$ and ${}_1F_1(\cdot)$. The functions ${}_2F_1(\cdot)$ and ${}_1F_1(\cdot)$ are the special cases of the well-known generalized hypergeometric series

$${}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{matrix} z\right] = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n} \frac{z^n}{n!},$$

where $(\lambda)_n$ is the Pochhammer symbol defined for $\lambda \in \mathbb{C}$ by

$$(\lambda)_n = \begin{cases} 1, & n = 0; \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & n \in \mathbb{N}. \end{cases}$$

Let us consider the function

$${}_pF_{q+r}^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1,\ldots,x_p;\\ y_1,\ldots,y_{q+r};\end{matrix} z;b\right] = {}_1F_r\left[\begin{matrix}1;\\ y_1,\ldots,y_r;\end{matrix} z;b\right] {}_pF_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}x_1,\ldots,x_p;\\ y_{1+r},\ldots,y_{q+r};\end{matrix} z;b\right]. \tag{1.3}$$

Prabhakar and Rekha [3] defined the polynomials

$$L_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha n+\beta+1)}{\Gamma(n+1)} \sum_{k=0}^{n} \frac{(-n)_k x^k}{k!\,\Gamma(\alpha k+\beta+1)} \tag{1.4}$$

for $\alpha \in \mathbb{C}^+$, $\beta \in \mathbb{C}^+_{-1}$, and $n \in \mathbb{N}$. When $\alpha = 1$, Eq. (1.4) reduces to

$$L_n^{(1,\beta)}(x) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \sum_{k=0}^{n} \frac{(-n)_k x^k}{k!\,\Gamma(k+\beta+1)} = L_n^{\beta}(x),$$

where $L_n^{\beta}(x)$ is the well-known generalized Laguerre polynomials. See [4].

The Konhauser polynomials of second kind is defined in [5] as

$$Z_n^{\beta}(x;k) = \frac{\Gamma(kn+\beta+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\beta+1)} \tag{1.5}$$

for $\beta \in \mathbb{C}^+_{-1}$, $n \in \mathbb{N}$, and $k \in \mathbb{Z}$.

It can be easily verified that

$$L_n^{(k,\beta)}\left(x^k\right) = Z_n^{\beta}(x;k)$$

and

$$L_n^{\beta}(x) = Z_n^{\beta}(x;1). \tag{1.6}$$

The polynomials $Z_n^{(\alpha,\beta)}(x;k)$ are defined [6] as

$$Z_n^{(\alpha,\beta)}(x;k) = \sum_{j=0}^{n} \frac{\Gamma(kn+\beta+1)(-1)^j x^{kj}}{j!\,\Gamma(kj+\beta+1)\,\Gamma(\alpha n-\alpha j+1)} \tag{1.7}$$

for $\alpha \in \mathbb{C}^+$, $\beta \in \mathbb{C}^+_{-1}$, $n \in \mathbb{N}$, and $k \in \mathbb{Z}$.

From (1.5) and (1.7), we acquire

$$Z_n^{(1,\beta)}(x;k) = Z_n^{\beta}(x;k).$$